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Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2018

Second Year

MATHEMATICS

Topology and Functional Analysis

Time : 3 Hours **Maximum Marks :70**

Answer any five questions.
Selecting atleast two from each section. All questions carry equal marks.

SECTION - A

Q1) a) Define topological space and give an example.

Let τ_1 and τ_2 be two topologies on a nonempty set X. Show that $\tau_1 \cap \tau_2$ is also a topology on X.

b) Let $f : X \to Y$ be a mapping of one topological space into another. Show that *f* is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

Q2) a) Prove that any closed subspace of a compact space is compact.

b) State and prove Heine – Borel theorem.

Q3) a) Prove that a metric space is sequentially compact if and only if it has the Bolzano – Weierstrass property.

b) Prove that a metric space is compact if and only if it is complete and totally bounded.

Q4) a) Prove that in a Hausdorff space, any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighborhoods.

b) State and prove Tietze extension theorem.

Q5) a) Prove that a subspace of the real line $\mathbf{R}$ is connected if and only if it is an interval.

b) Let X be a topological space and A a connected subspace of X. Prove that if B is a subspace X such that $A \subseteq B \subseteq \overline{A}$ then B is connected. In particular, show that $\overline{A}$ is connected.

SECTION - B

Q6) a) If N and N' are normed linear spaces then prove that the set $\mathcal{B}$ (N, N′) of all continuous linear transformations of N into N' is itself a normed linear space with respect to the pointwise linear operations and the norm defined by $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$.

Further, show that if N' is a Banach space, then prove that $\mathcal{B}$ (N, N′) is also a Banach space.

b) i) If M is a closed linear subspace of a normed linear space N and X_0 is a vector not in M, then prove that there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

ii) If N is a normed linear space then prove that there exists a natural imbedding of N in N^{**}.

Q7) a) State and prove the closed graph theorem.

b) State and prove the uniform boundedness theorem.

Q8) a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

b) Prove that if M is a closed linear subspace of a Hilbert space H then prove that $H = M \oplus M^{\perp}$.

Q9) a) State and prove Gram-Schmidt orthogonalization process.

b) Define the adjoint of an operator T on a Hilbert space H and prove that the mapping $T \mapsto T^*$ is a mapping of $\mathcal{B}(H)$ into itself.

Q10) a) Define a projection and a perpendicular projection. Let M be a closed linear subspace of a Hilbert space H. Then prove that P is the projection on M if and only if I-P is the projection on $M^{\perp}$. Also, prove that if P is a projection on M then

$$x \in M \Leftrightarrow Px = x \Leftrightarrow \|Px\| = \|x\|$$

b) If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of a Helbert space H then prove that $P = P_1 + P_2 + \ldots + P_n$ is a projection if and only if the P_i's are pairwise orthogonal; and in this case, P is the projection on $M = M_1 + M_2 + \ldots + M_n$.

(DM22)

Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2018

Second Year

MATHEMATICS

Measure and Integration

Time : 3 Hours **Maximum Marks :70**

Answer any five questions.
All questions carry equal marks.

Q1) a) Let A be a countable set. Prove that the set of all finite sequences from A is also countable.

b) State axiom of choice and apply it to prove the following:
"Let $f : \mathrm{X} \to \mathrm{Y}$ be a mapping on to Y. Then there is a mapping $g : \mathrm{Y} \to \mathrm{X}$ such that *fog* is the identity map on Y".

Q2) a) Prove that the interval (a, ∞) is measurable.

b) Prove that the collection $\mathcal{M}$ of measurable sets is a σ – algebra.

Q3) a) Let C be a constant and f and g two measurable real – valued functions defined on the same domain. Then prove that the functions $f + c$, cf, $f + g$, $g - f$, and fg are also measurable.

b) Let E be a measurable set of finite measure, and $\{f_n\}$ a sequence of measurable functions defined on E. Let f be a real – valued function such that for each x in E we have $f_n(x) \to f(x)$. Then prove that given $\in > 0$ and $\delta > 0$ there is a measurable set $\mathrm{A} \subset \mathrm{E}$ with $m(\mathrm{A}) < \delta$ and an integer N such that for all $x \notin \mathrm{A}$ and all $n \geq \mathrm{N}$, we have $| f_n(x) - f(x) | < \in$.

Q4) a) Let φ and ψ be simple functions which vanish outside a set of finite measure. Then prove that $\int (a\varphi + b\psi) = a\int \varphi + b\int \psi$, and if $\varphi \geq \psi$ a.e. then $\int \varphi \geq \int \psi$.

b) Let f be a nonnegative function which is integrable over a set E. Then prove that given $\in > 0$ there is a $\delta > 0$ such that for every set $\mathrm{A} \subset \mathrm{E}$ with $m(\mathrm{A}) < \delta$ we have $\int_{\mathrm{A}} f < \in$.

Q5) a) State and prove Lebesgue convergences theorem.

b) Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \le \int_E |f|$. Does the integrability of $|f|$ imply that of f?

Q6) a) State and prove Vitali lemma.

b) Prove that if f is bounded and measurable on $[a, b]$ and $F(x) = \int_a^x f(t)dt + F(a)$, then $F'(x) = f(x)$ for almost all x in $[a, b]$.

Q7) a) State and prove Holder inequality.

b) Prove that a normed linear space X is complete if and only if every absolutely summable series summable.

Q8) a) Let $(X, \mathcal{B}, \mu)$ be a measure space. Then prove that there exists a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that

i) $\mathcal{B} \subset \mathcal{B}_0$

ii) $E \in \mathcal{B} \Rightarrow \mu(E) = \mu_0(E)$, and

iii) $E \in \mathcal{B}_0 \Leftrightarrow E = A \cup B$ where $B \in \mathcal{B}$ and $A \subset C, C \in \mathcal{B}$ with $\mu(C) = 0$.

b) Let υ be a signed measure on the measurable space $(X, \mathcal{B})$. Then prove that there are two mutually singular measures υ^+ and υ^- on $(X, \mathcal{B})$ such that $\upsilon = \upsilon^+ - \upsilon^-$. Moreover, show that there is only one such pain of mutually singular measures.

Q9) State and prove the Radon – Nikodym theorem.

Q10) a) Define measure on an algebra $\mathcal{A}$ and prove that if $A \in \mathcal{A}$ then A is measurable with respect to μ^*.

b) State and prove caratheodory extension theorem.

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(DM23)

Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2018

Second Year

MATHEMATICS

Analytical Number Theory and Graph Theory

Time : 3 Hours **Maximum Marks :70**

Answer any five of the following questions, selecting atleast two questions from each section.

All questions carry equal marks.

SECTION - A

Q1) a) State and prove Euler's summation formula.

b) For all $x \geq 1$ show that $\sum_{n \leq x} d(n) = x \log x + (2C-1)x + O(\sqrt{x})$ for C the Euler's constant.

Q2) a) For $x > 1$ show that $\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$.

b) For all $x \geq 1$ show that $\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1$ with equality holding only if $x < 2$.

Q3) a) State and prove Abel's identity.

b) For $x \geq 2$ show that $\theta(x) = \Pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$.

Q4) a) Let F be a real or complex valued function defined on $(0, \infty)$, and $G(x) = \log x . \sum_{n \leq x} F\left(\frac{x}{n}\right)$ then show that $F(x) \log x + \sum_{n \leq x} F\left(\frac{x}{n}\right) \wedge (n) = \sum_{d \leq x} \mu(d) G(\frac{x}{d})$.

b) State and prove Selberg's asymptotic formula.

SECTION – B

Q5) a) If a graph has exactly two vertices of odd degree show that there is a path joining these vertices.

b) Show that a connected graph G is an Euler graph if and only if it can be decomposed into circuits.

Q6) a) Show that in a complete graph with n vertices there are (n-1)/2 edge – disjoint Hamiltonian circuits if n is an odd number such that $n \geq 3$.

b) Draw a graph that has a Hamiltonian path but does not have a Hamiltonian circuit.

Q7) a) Show a graph with n vertices, (n-1) edges and no circuits is connected.

b) Show that every tree has either one or two centers.

Q8) a) Show that every circuit has an even number of edges in common with any cut-set.

b) Show that ν in a connected graph G is a cut vertex if and only if there exist two vertices x and y in G such that every path between x and y passes through ν.

Q9) Show that the complete graph of five vertices is non planar.

Q10) a) Show that the ring sum of two circuits in a graph G is either a circuit or an edge disjoint union of circuit.

b) Write a note on the vector space associated with a graph G.

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(DM24)

Total No. of Questions :10] **[Total No. of Pages : 02**

M.Sc. (Second) DEGREE EXAMINATION, MAY– 2018

Second Year

MATHEMATICS

Rings and Modules

Time :3 Hours **Maximum Marks :70**

Answer any five questions.

All questions carry equal marks.

Q1) a) Prove that a Boolean algebra becomes a complemented distributive lattice by defining $a \vee b = (a' \wedge b')', 1 = 0'$.

b) If θ is a reflexivehomomorphic relation on a ring, then prove that θ is symmetric and transitive, hence a congruence relation.

Q2) a) If A and B are ideals of a ring R then prove that AB, $A_{\bullet}{}^{\bullet}B$ and $A^{\bullet}{}_{\bullet}B$ are ideals of R. Moreover, prove that

i) AR = A=RA

ii) $A_{\bullet}{}^{\bullet}R$=A= $R^{\bullet}{}_{\bullet}A$

iii) $A_{\bullet}{}^{\bullet}A$ = R = $A^{\bullet}{}_{\bullet}A$

iv) AB $\subset$ A $\cap$ B

b) Prove that every proper right ideal in a ring is contained in a maximal proper right ideal.

Q3) a) Verify that Hom_R(A, B) is an Abelian group.

b) Prove that a module has a composition series if and only if it is both Artinian and Noetherian.

Q4) a) Prove that a proper ideal M of the commutative ring R is maximalif and only if $\forall_{r \notin M} \exists_{n \in R} 1 - rn \in M$

b) Prove that every ring is a sub direct product of sub directly irreducible rings.

Q5) a) Let B be a Boolean algebra and K be an ideal of B. Then prove that K is maximal if and only if K is a prime ideal.

b) If R is any commutative ring, then prove that $Q(R)$ is rationally complete.

Q6) a) If A_R is an irreducible module, then prove that its ring of endomorphisms $D = Hom_R(A, A)$ is a division ring.

b) Prove that R is a Prime ring if and only if $1 \neq 0$ and, for all $\alpha \neq 0$ and $b \neq 0$ in R, there exists $r \in R$ such that $arb \neq 0$.

Q7) a) Prove that the Prime radical of R is the set of all strongly nilpotent elements.

b) Prove that R is semiprimitive if and only if it is a sub direct product of primitive rings.

Q8) a) If R is semiprime and $e^2 = e \in R$, then prove that eR is a minimal right ideal if and only if eRe is a division ring.

b) Prove that the radical of a right Artinian ring is nilpotent.

Q9) a) Prove that M_R is free if and only if it is isomorphic to a direct sum of copies of R_R.

b) Prove that every module is isomorphic to a factor of a free module.

Q10) a) Prove that an Abelian group is injective if and only if it is divisible.

b) If ${}_R F$ is a free module then prove that F_R^* is injective.

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