(DM01)

Total No. of Questions :10] **[Total No. of Pages : 02**

M.Sc. (Previous) DEGREE EXAMINATION, MAY– 2018

First Year

MATHEMATICS

Algebra

Time :3 Hours **Maximum Marks :70**

SECTION - A

Answer any five questions.

All questions carry equal marks.

Q1) a) State and prove Sylow's theorem for abelian groups.

b) If G is a group then show that A(G) the set of automorphisms of G is also a group.

Q2) a) Show that every group is isomorphic to a subgroup of A(s) for some appropriate S.

b) Show that conjugacy is an equivalence relation onG.

Q3) a) Let G be a group and if G in the internal direct product of N_1 N_2 … N_n then show that G is isomorphic to $N_1 \times N_2 \times \dots \times N_n$.

b) Describe all finite abelian groups of order $2^4\ 3^4$.

Q4) a) Show that a finite integral domain is a field.

b) If R is a commutative ring with unit element and M is an ideal R, then show that M is a maximal ideal of R if and only if R/M is a field.

Q5) Show that every integral domain can be imbedded in a field.

Q6) a) If L is a finite extension of K and if K is a finite extension of F, then show that L is a finite extension of F in particular [L:F] = [L:K] [K:F].

b) If P(x) is a polynomial in F(x) of degree $n \geq 1$ and irreducible over F then show that there is an extension E of F such that [E:F] = n in which P(x) has a root.

Q7) a) Show that the polynomial f(x) EF[x] has a multiple root if and only if f(x) and $f'(x)$ have a nontrivial common root.

b) If K is finite extension of F, then show that G (K, F) is a finite group with its order O(G, F) satisfie $O(G(K,F)) \leq [K:F]$.

Q8) a) Show that a group G is solvable if and only if $G^{(K)} = e$ for some integer k.

b) Show that the general polynomial $P(x) = x^n + a_1 x^n + \ldots + a_n$ for n≥5 is not solvable by radicals.

Q9) a) Show that a lattice of invariant subgroups of any group is modular.

b) If a and$b$ are any two elements of a modular lattice then show that the intervals I $[a \cup b, a]$ and I $[b, a \cap b]$ are isomorphic.

Q10)Show that if L is a complemented modular lattice that satisfies both chain conditions, then the element 1 of L is a $1 \cup b$ of independent points and conversely if L is a modular lattice with 0 and 1 such that 1 is a $1 \cup b$ of a finite number of points then L is complemented and satisfies both chain conditions.

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(DM02)

Total No. of Questions : 10] **[Total No. of Pages : 03**

M.Sc. (Previous) DEGREE EXAMINATION, MAY – 2018

First Year

MATHEMATICS

Analysis

Time : 3 Hours **Maximum Marks :70**

Answer any five of the following questions.
All questions carry equal marks.

Q1) a) Let A be the set of all sequences whose elements are the digits 0 and 1. Prove that this set A is uncountable.

b) Prove that every k-cell is compact.

Q2) a) Prove that if a set E in $\mathbb{R}^k$ has one of the following three properties, then it has the other two:

i. E is closed and bounded.

ii. E is compact.

iii. Every infinite subset of E has a limit point in E.

b) Prove that a subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property:
If $x \in \mathrm{E}, y \in \mathrm{E}$, and $x < z < y$, then $z \in \mathrm{E}$.

Q3) a) Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and that e is irrational.

b) Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty .$$

Then prove that there exists a rearrangement $\sum a_n'$ with partial sums s_n' such that $\liminf_{n\to\infty} s_n' = \alpha, \quad \limsup_{n\to\infty} s_n' = \beta$.

Q4) a) Let f be a continuous mapping of a compact metric space X into a metric space Y. Then prove that f is uniformly continuous on X.

b) Let E be a nonempty subset of a metric space X, define the distance from x in X to E by

$$P_E(x) = \inf_{z \in E} d(x, z)$$

i) Prove that $P_E(x) = 0$ if and only if $x \in \overline{E}$.

ii) Prove that P_E is a uniformly continuous function on X, by showing that $|P_E(x) - P_E(y)| \le d(x, y)$ for all $x \in X, y \in X$.

Q5) a) Define Riemann – Stieltjes integral. Prove that if f is bounded on $[a, b]$, f has only finitely many points of discontinuity on [a, b], and α is continuous at every point at which f is discontinuous then $f \in R(\alpha)$.

b) Suppose $f \ge 0, f$ is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Q6) a) Suppose $c_n \ge 0$ for n = 1, 2, …, Σc_n converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$, where I is the unit step function. Let f be continuous on $[a, b]$ then prove that $\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

b) Assume that $f(x) \ge 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^{\infty} f(x)dx$ converges if and only $\sum_{n=1}^{\infty} f(n)$ converges.

Q7) a) If $\{f_n\}$ is a sequence of continuous functions on a subset E of a metric space X, and if $f_n \to f$ uniformly on E then prove that f is continuous on E.

b) Suppose f_n is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$ then prove that $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n \to \infty} f_n'(x), a \le x \le b$.

Q8) a) Prove that if $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every x in E.

b) State and prove Weierstrass approximation theorem.

Q9) a) Let f and g be measurable real-valued functions defined on the measurable space X, let F be a real and continuous on $\mathbb{R}^2$, and put $h(x) = \mathrm{F}(f(x), g(x)),\ x \in \mathrm{X}$. Then prove that h is measurable.

b) State and prove Lebesgue's monotone convergence theorem.

Q10) a) State and prove Fatou's theorem.

b) Prove that $L^2(\mu)$ is a complete metric space.

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(DM03)

Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. (Previous) DEGREE EXAMINATION, MAY – 2018

First Year

MATHEMATICS

Complex Analysis & Special Functions & Partial Differential Equations

Time : 3 Hours **Maximum Marks :70**

Answer any five questions choosing at least two from each section.

All questions carry equal marks.

SECTION - A

Q1) a) Find a power series solution of the Legendre's equation
$(1-x^2)y''-2xy'+n(n+1)y=0$

b) State and prove Laplace's first and second integrals for $P_n(x)$.

Q2) a) Prove that $\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx=\frac{2n}{4n^2-1}$.

b) Using Rodrigue's formula, find the values of $P_0(x)$, $P_1(x)$, $P_2(x)$ and $P_3(x)$.

Q3) a) Prove that $\frac{d}{dx}\{xJ_n(x)J_{n+1}(x)\} = J_n^2(x)-J_{n+1}^2(x)$.

b) Solve $(yz + xyz)\, dx + (zx + xyz)\, dy + (xy + xyz)dz = 0$.

Q4) a) Find the general solution of $\left(D^2+DD'+D'-1\right)z=\sin\left(x+2y\right)$.

b) Solve $\left(D^2+2DD'+\left(D'\right)^2\right)z=e^{2x+3y}$.

Q5) a) Solve $\left(D^2-D'\right)z=2y-x^2$.

b) Solve $(r-s)x = (t-s)y$ by using Monge's method.

SECTION - B

Q6) a) Calculate the n^{th} roots of unity and deduce the cube roots of unity.

b) Prove that if G is open and connected and f : G $\rightarrow \mathbb{C}$ is differentiable with $f'(z)=0$ for all z in G, then f is constant.

Q7) a) Prove that if $\gamma : [a, b] \to \mathbb{C}$ is piecewise smooth then γ is of bounded variation and $V(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

b) State and prove the fundamental theorem of algebra.

Q8) a) State and prove Cauchy's integral formula, first version.

b) Let G be an open set and let f : G $\to \mathbb{C}$ be a differentiable function. Then prove that f is analytic on G.

Q9) a) State and prove Casorati – Weierstrass theorem.

b) Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$; give the Laurent Expansion of $f(z)$ in each of the following annuli :

i) ann (0; 0, 1);

ii) ann (0; 1, 2);

iii) ann (0; 2, ∞).

Q10) a) State and prove residue theorem.

b) Show that $\int_0^\infty \dfrac{\sin x}{x} dx = \dfrac{\Pi}{2}$.

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(DM04)

Total No. of Questions : 10] [Total No. of Pages : 03

M.Sc. (Previous) DEGREE EXAMINATION, MAY – 2018

First Year

MATHEMATICS

Theory of Ordinary Differential Equations

Time : 3 Hours **Maximum Marks :70**

Answer any five questions.
All questions carry equal marks.

Q1) a) Let $a_1,...,a_n$ be continuous functions on an interval I. Prove that there exist n linearly independent solutions of $\mathrm{L}(y) \equiv y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y = 0$ on I.

b) Consider the equation $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$ for $x > 0$.

i) Show that there is a solution of the form x^r, where r is a constant.

ii) Find two linearly independent solutions for $x > 0$, and prove that they are linearly independent.

iii) Find the two solutions ϕ_1, ϕ_2 satisfying

$\phi_1(1) = 1, \phi_2(1) = 0,$

$\phi_1'(1) = 0, \phi_2'(1) = 1,$

Q2) a) Find all solutions of the equation $y'' - \frac{2}{x^2}y = x, 0 < x < \infty$.

b) Find two linearly independent power series solutions (in powers of x) of the differential equation $y'' - xy = 0$ on $-\infty < x < \infty$.

Q3) a) Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle.

R: $|x - x_0| \le a, |y - y_0| \le b$.

Then prove that the equation $\mathrm{M}(x, y) + \mathrm{N}(x, y)\ y' = 0$

is exact in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R.

b) Compute the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$ of the equation $y' = xy, y(0) = 1$. Also compute the solution.

Q4) a) Define Lipschitz condition. Suppose S is either a rectangle $|x-x_0|\le a, |y-y_0|\le b, a,b>0$ or a strip $|x-x_0|\le a, |y|<\infty, a>0$ and that f is a real-valued function defined on S such that $\frac{\partial f}{\partial y}$ exists, is continuous on S, and $|\frac{\partial f}{\partial y}(x-y)|\le K,\ (x,y)$ in S.

for some K > 0. Then prove that f satisfies a Lipschitz condition on S with Lipschitz constant K.

b) Let $f(x,y)=\frac{\cos y}{1-x^2},\ |x|<1$.

i) Show that f satisfies a Lipschitz condition on every strip $S_a:|x|\le a$ where $0<a<1$.

ii) Show that every initial value problem $y'=f(x,y), y(0)=y_0, |y_0|<\infty$ has a solution which exists for $|x|<1$.

Q5) a) Give an example of a system of differential equations which arise in the study of dynamics of central forces and planetary motion.

b) Find a solution ϕ of $y''=-\frac{1}{2y^2}$ satisfying $\phi(o)=1, \phi'(o)=-1$.

Q6) a) Let $\overline{f}$ be the vector – valued function defined on $R:|x|\le 1, |\overline{y}|\le 1\ (\overline{y}\text{ in } C_2)$ by $\overline{f}(x,\overline{y})=(y_2^2+1,\ x+y_1^2)$.

i) Find an upper bound M for $\left|\overline{f}(x,\overline{y})\right|$ for $(x,\overline{y})$ in R.

ii) Compute a Lipschitz constant K for $\overline{f}$ on R.

b) Consider the system

$$y_1'=3y_1+xy_3$$
$$y_2'=y_2+x^3y_3$$
$$y_3'=2xy_1-y_2+e^xy_3.$$

Show that every initial value problem for this system has a unique solution which exists for all real x.

Q7) a) Find functions $z(x)$, $k(x)$ and $m(x)$ such that

$$z(x)\left[x^2y''-2xy'+2y\right]=\frac{d}{dx}\left[k(x)y'+m(x)y\right]$$

and hence solve

$$x^2y''-2xy'+2y=0, x>0.$$

b) Show that if z, z_1, z_2 and z_3 are any four different solutions of the Riccati equation.

$z^1 + a(x)z + b(x)z^2 + c(x) = 0$

then show that

$\frac{z - z_2}{z - z_1} = \frac{z_3 - z_1}{z_3 - z_2} =$ constant.

Q8) a) Find the general solution of $y'' - 3y' + 2y = f(x), -\infty < x < \infty$ where f is a continuous function and then evaluate the general solution when $f(x) = x$.

b) Given the differential equation $4x^2 y'' + y = f(x),\ 1 \le x < \infty$ compute Green's function and then compute particular solution. Also, find the general solution when $f(x) = x$.

Q9) a) State and prove Sturm separation theorem.

b) Discuss the oscillations of the Bessel equation

$x^2 y'' - xy' + (x^2 - n^2)y = 0,$

Where n is a constant.

Q10) a) Solve

$x^2 y'' - 2xy' + (2 + x^2)y = 0, x > 0$

b) State and prove Gronwall's inequality.

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