INTEGRAL EQUATIONS

M.Sc., MATHEMATICS First Year

Semester – II, Paper-IV

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M.Sc., MATHEMATICS – INTEGRAL EQUATIONS

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FOREWORD

Since its establishment in 1976, Acharya Nagarjuna University has been forging ahead in the path of progress and dynamism, offering a variety of courses and research contributions. I am extremely happy that by gaining 'A⁺' grade from the NAAC in the year 2024, Acharya Nagarjuna University is offering educational opportunities at the UG, PG levels apart from research degrees to students from over 221 affiliated colleges spread over the two districts of Guntur and Prakasam.

The University has also started the Centre for Distance Education in 2003-04 with the aim of taking higher education to the doorstep of all the sectors of the society. The centre will be a great help to those who cannot join in colleges, those who cannot afford the exorbitant fees as regular students, and even to housewives desirous of pursuing higher studies. Acharya Nagarjuna University has started offering B.Sc., B.A., B.B.A., and B.Com courses at the Degree level and M.A., M.Com., M.Sc., M.B.A., and L.L.M., courses at the PG level from the academic year 2003-2004 onwards.

To facilitate easier understanding by students studying through the distance mode, these self-instruction materials have been prepared by eminent and experienced teachers. The lessons have been drafted with great care and expertise in the stipulated time by these teachers. Constructive ideas and scholarly suggestions are welcome from students and teachers involved respectively. Such ideas will be incorporated for the greater efficacy of this distance mode of education. For clarification of doubts and feedback, weekly classes and contact classes will be arranged at the UG and PG levels respectively.

It is my aim that students getting higher education through the Centre for Distance Education should improve their qualification, have better employment opportunities and in turn be part of country's progress. It is my fond desire that in the years to come, the Centre for Distance Education will go from strength to strength in the form of new courses and by catering to larger number of people. My congratulations to all the Directors, Academic Coordinators, Editors and Lesson-writers of the Centre who have helped in these endeavors.

Prof. K.GangadharaRao
M.Tech.,Ph.D.,
Vice-Chancellor I/c
Acharya Nagarjuna University

M.Sc. – Mathematics Syllabus

SEMESTER-II

204MA24-INTEGRAL EQUATTONS

UNIT-I

Volterra Integral Equations: Basic Concepts - Relationship between linear differential equations and volterra lntegral equations - Resolvent Kemel of Volterra lntegral Equation- Solution of Integral Equation by Resolvent Kemel - Method of Successive Approximations – Convolution type equations.(Sections 1 to 5 of Chapter -I of the Reference Book)

UNIT-II

Solution of Integro differential equations with the Aid of the Laplace Transformation – volterre Integral Equations with limits $(x, +\infty)$-Volterra Integral Equations of the First Kind – Euler Integrals - Abels problem, Ables Integral Equations and its generalizations - Volterra Integral equations of the First kind of the Convolution type. (Sections 6 to 11 of Chapter - I of the reference book).

UNIT-III

Fredholm Integral Equations: Fredholm equations of the Second kind' Fundamentals -the method of Fredholm determinants - Iterated Kernels. Constructing the Resolvent Kernel with the Aid of iterated Kernels - Integral equations with degenerate kernel - Characteristic Numbers and Eigenfunctions. (Sections 12 to 16 of Chapter -II of Reference Book).

UNIT-IV

Solution of Homogeneous Integral Equations with Degenerate Kernel -Nonhomogeneous Symmetric Equations - Fredholm Altemative - Constuction of Green's Function for Ordinary Differential Equations - Using Green's Function if the Solution of Boundary Value Problems. (Sections 17 to 2l of Chapter -II of the Reference Book).

UNIT -V

Boundary value problems containing a parameter reducing them to Integral equations – singular Integral equations - Approximate methods of solving Integral equations' (sections 22,23, of Chapter -II and 24 of Chapter-Ill of Reference Book)

Reference Book: Problems and Exercises in Integral Equations, MIR Oybkusgers, Moscow 197I by M. Krsnov, A. Kiselev and G' Makarendo.

Text Books:

l. Integral Equations and their Applications, WIT press, 25 Bridge Street,Billerica, MA 01821, USA, by M. Rahman.

2. Introduction to Integral Equations with Applications, John wiley & Sons,1999, by Jeni, A.

3. Linear Integral Equation, Theory and Techniques, Academic Press' 20l4 by Kanwal R, P.

4. A first course in Integral Equations, 2nd edition, world scientific Publishing Co. 2015 by Wazwaz ,A. M

CODE:204MA24

M.Sc DEGREE EXAMINATION

Second Semester Mathematics :: Paper IV-INTEGRAL EQUATIONS

MODEL QUESTION PAPER

Time: Three hours Maximum:70 Marks

Answer ONE question from each unit (5x14=70)

UNIT-I

1. (a) Form an integral equation corresponding to the differential equation

$$y'' + xy' + y = 0$$

with the initial conditions, $y(0) = 1, y'(0) = 0$

(b) Find the resolvent kernel of the Volterra integral equation with kernel $K(x,t) = x - t$.

Or

2. (a) Using the method of successive approximations, solve the integral

$$\varphi(x) = 1 + \int_0^x \varphi(t)\, dt \text{ , taking } \varphi_0(x) = 0.$$

(b) Solve the integral equation,

$$\varphi(x) = \sin x + 2\int_0^x \cos(x-t)\varphi(t)\ dt.$$

UNIT-II

3. (a) Solve the integro-differential equation,

$$\varphi''(x) + \varphi(x) + \int_0^x \sinh(x-t)\,\varphi(t)dt + \int_0^x \cosh(x-t)\,\varphi'(t)dt = coshx;$$

$\varphi(0) = -1,\ \varphi'(0) = 1$, by using the Laplace Transformation.

(b) Solve the integral equation, $\varphi(x) = cosx + \int_x^\infty e^{(x-t)}\varphi(t)dt.$

Or

4. (a) Solve, $\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = x^{\frac{1}{2}}$

(b) Solve, $2\varphi(x) - \int_0^x \varphi(t)\varphi(x-t)dt = sinx$

UNIT-III

5. (a) Show that the function $\varphi(x)=\sin\frac{\pi x}{2}$ is a solution of the Fredholm-type integral equation, $\varphi(x)-\frac{\pi^2}{4}\int_0^1 K(x,t)\varphi(t)dt=\frac{x}{2}$.

(b) Find the iterated kernels of the following kernel for specified *a* and *b*.

$$K(x, t)=e^x\cos t;\ a=0,\ b=\pi.$$

Or

6. (a) Solve the given integral equation with a degenerate kernel,

$$\varphi(x)-\lambda \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan t\, \varphi(t)\, dt = \cot x$$

(b) Find the eigenfunction and the corresponding characteristic numbers of the equation, $\varphi(x) = \lambda \int_{-\pi}^{\pi} \cos^2(x-t)\, \varphi(t)\, dt$.

UNIT-IV

7. (a) Solve the following homogeneous integral equation:

$$\varphi(x)+6 \int_0^1 (x^2-2xt)\, \varphi(t)\, dt=0.$$

(b) Solve the homogeneous symmetric integral equation:

$$\varphi(x) + \int_0^1 K(x,t)\, \varphi(t)\, dt = xe^x,$$

$$K(x,t) = \begin{cases} \dfrac{\sinh x \sinh(t-1)}{\sinh 1}, & 0 \le x \le t, \\ \dfrac{\sinh t \sinh(x-1)}{\sinh 1}, & t \le x \le 1. \end{cases}$$

Or

8. Construct Green's function for the homogeneous boundary value problem $y^{IV}(x) = 0$,

$$\left.\begin{array}{l} y(0) = y'(0) = 0 \\ y(1) = y'(1) = 0 \end{array}\right\}.$$

UNIT-V

9. (a) Reduce the boundary value problem,

$y'' + \lambda y = x, \quad y(0) = y\left(\frac{\pi}{2}\right) = 0$ to an integral equation.

(b) Show that the integral equation $\varphi(x) = \lambda \int_0^{\infty} J_v\left(2\sqrt{xt}\right)\varphi(t)dt$ has characteristic number $\lambda = \pm 1$ of infinite multiplicity and find the associated eigenfunctions. [where $J_v(z)$ is a Bessel function of the first kind.]

Or

10. (a) Solve the integral equation $\frac{1}{\sqrt{\pi x}} \int_0^{\infty} e^{\frac{-t^2}{4x}} \varphi(t)dt = 2x - sinhx.$

(b) Use the Bubnov-Galerkin method to solve the equation

$$\varphi(x) = x + \int_{-1}^{1} x\, \varphi(t)dt .$$

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3.	Solution of Volterra-Type Integral Equation by Using the Method of Successive Approximations	3.1 – 3.7
4.	Solution of Volterra-Type Integral Equation by Using Convolution Theorem	4.1 – 4.8
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LESSON - 1
VOLTERRA INTEGRAL EQUATIONS

OBJECTIVES:

- To identify the Integral equations
- To classify the types of integral equations
- To verify that the given function is a solution of the integral equation
- To convert the given Initial value problem and Boundary value problem to an equivalent integral equation

STRUCTURE:

1.1 Introduction

1.2 Integral Equations

1.3 Linear and Non-linear Integral Equations

1.4 Classification of Linear Integral Equations

1.5 Solution of the Integral Equation

1.6 Solved Examples

1.7 Differentiation of a Function Under an Integral Sign

1.8 Relationship Between Linear Differential Equations and Volterra Integral Equations

1.9 Summary

1.10 Technical Terms

1.11 Self-Assessment Questions

1.12 Suggested Readings

1.1 INTRODUCTION:

Integral equations arise in the modeling of physical situations in science, technology, and engineering. They also arise as representation formulae for the solution of differential equations. There is a relation between the solutions of initial value problems and boundary value problems; differential equations with initial and boundary conditions can be equivalently represented as integral equations.

1.2 INTEGRAL EQUATIONS:

An integral equation is an equation in which an unknown function, to be determined, appears under one or more integral signs.

For example, for $a \leq x \leq b, a \leq t \leq b$, the equations

$$\int_a^b K(x,t)\varphi(t)dt = f(x), \qquad (1.1)$$

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x) \qquad (1.2)$$

and

$$\varphi(x) = \int_a^b K(x,t)[\varphi(t)]^2 \, dt, \qquad (1.3)$$

where the function $\varphi(x)$, is the unknown function, while the functions $f(x)$ and $K(x,t)$ are known functions and $\lambda, a,$ and b are constants, are all integral equations. These functions may be complex-valued functions of the real variables x and t.

1.3 LINEAR AND NON-LINEAR INTEGRAL EQUATIONS:

An integral equation is called linear if only linear operations are performed in it upon the unknown function. An integral equation that is not linear is known as a non-linear integral equation. By writing either

$$L(\varphi) = \int_a^b K(x,t)\varphi(t)dt \; (or) \; L(\varphi) = \varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt$$

we can easily verify that L is a linear operator. In fact, for any constants c_1 and c_2, we have:

$$L\{c_1\varphi_1(x) + c_2\varphi_2(x)\} = c_1 L\{\varphi_1(x)\} + c_2 L\{\varphi_2(x)\},$$

which is a well-known general criterion for a linear operator. For example, the integral equations (1.1) and (1.2) of Section 1.2 are linear integral equations, while the integral equation (1.3) is a non-linear integral equation. The most general type of linear integral equation is of the form:

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_\Omega K(x,t)\varphi(t)dt, \qquad (1.4)$$

where the upper limit may be either variable x or constant. The functions $\alpha(x), f(x)$ and $K(x,t)$ are known functions while $\varphi(x)$ is to be determined; λ is a non-zero real or complex parameter. The function $K(x,t)$ is known as the kernel of the integral equation.

1.4 CLASSIFICATION OF LINEAR INTEGRAL EQUATIONS:

1.4.1 Volterra Integral Equation:

An integral equation is said to be a Volterra integral equation if the upper limit of integration is a variable x.

The general form is:

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\varphi(t)dt.$$

(i) When $\alpha \equiv 0$, the equation involves the unknown function φ appearing only under the integral sign and nowhere else in the equation, then

$$f(x) = -\lambda \int_a^x K(x,t)\varphi(t)dt\,, \qquad a > -\infty,$$

is called the Volterra's integral equation of the first kind.

(ii) When $\alpha \equiv 1$, the equation involves the unknown function φ both inside and outside the integral sign, then

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\varphi(t)dt$$

is called the Volterra's integral equation of the second kind.

(iii) When $\alpha \equiv 1$ and $f(x) \equiv 0$, the equation reduces to

$$\varphi(x) = \lambda \int_a^x K(x,t)\varphi(t)dt$$

is called the homogeneous Volterra's integral equation of the second kind.

1.4.2 Fredholm Integral Equation:

An integral equation is said to be a Fredholm integral equation if the domain of integration Ω is fixed,

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt.$$

(i) When $\alpha \equiv 0$, the equation involves the unknown function φ only under the integral sign, then

$$f(x) = \lambda \int_a^b K(x,t)\varphi(t)dt\,, a \le x \le b$$

is called the Fredholm integral equation of first kind.

(ii) When $\alpha \equiv 1$, the equation involves the unknown function φ, both inside as well as outside the integral sign, then

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt\,, a \le x \le b$$

is called the non-homogeneous Fredholm integral equation of second kind.

(iii) When $\alpha \equiv 1, f(x) = 0$, the equation reduced to

$$\varphi(x) = \lambda \int_a^b K(x,t)\varphi(t)dt\,, a \le x \le b$$

called the homogeneous Fredholm integral equation of the second kind.

In this lesson, we mainly focus on the Volterra integral equations.

1.5 SOLUTION OF AN INTEGRAL EQUATION:

Consider the linear Volterra integral equations:

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt \qquad (1.5)$$

and

$$\alpha(x)\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt. \qquad (1.6)$$

A solution of the integral equation (1.5) or (1.6) is a continuous function $\varphi(x)$, which, when substituted into the equation, reduces it to an identity (with respect to x).

1.6 SOLVED EXAMPLES:

Example 1.1 Show that the function

$$\varphi(x) = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

is a solution of the Volterra integral equation

$$\varphi(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} \ \varphi(t)dt.$$

Solution. Given that the integral equation is

$$\varphi(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} \ \varphi(t)dt. \qquad (1.7)$$

Also, given

$$\varphi(x) = (1+x^2)^{-\frac{3}{2}}. \qquad (1.8)$$

Then, RHS of (1.7)

$= \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} \ (1+t^2)^{-\frac{3}{2}} dt$, using (1.8)

$= \frac{1}{1+x^2} - \frac{1}{1+x^2} \int_0^x t\,(1+t^2)^{-\frac{3}{2}} dt$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^{x^2}(1+u)^{-\frac{3}{2}}\,\frac{1}{2}\,du \text{ (by putting } t^2 = u \text{ and } 2tdt = du)$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[(1+u)^{-\frac{1}{2}}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2}\left[\frac{1}{(1+x^2)^{\frac{1}{2}}}\right]$$

$$= \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$= \varphi(x)$. (by (1.8))

Hence, (1.8) is a solution of the given integral equation (1.7).

Example 1.2 Verify the given function

$$\varphi(x) = \frac{x}{(1+x^2)^{\frac{5}{2}}}$$

is the solution of the integral equation

$$\varphi(x) = \frac{3x + 2x^3}{3(1+x^2)^2} - \int_0^x \frac{(3x + 2x^3 - t)}{(1+x^2)^2}\,\varphi(t)dt\,.$$

Solution. Given integral equation is

$$\varphi(x) = \frac{3x + 2x^3}{3(1+x^2)^2} - \int_0^x \frac{(3x + 2x^3 - t)}{(1+x^2)^2}\,\varphi(t)dt. \qquad (1.9)$$

Also, given

$$\varphi(x) = \frac{x}{(1+x^2\,)^{\frac{5}{2}}}. \qquad (1.10)$$

Then RHS of (1.9) =

$$= \frac{3x + 2x^3}{3(1+x^2)^2} - \int_0^x \frac{(3x + 2x^3 - t)}{(1+x^2)^2}\left[\frac{t}{(1+t^2)^{\frac{5}{2}}}\right]dt, \qquad \text{using} \quad (1.10)$$

$$= \frac{3x + 2x^3}{3(1+x^2)^2} - \frac{1}{2}\frac{3x + 2x^3}{(1+x^2)^2}\int_0^x \frac{2t}{\left((1+t^2)^{\frac{5}{2}}\right)}\,dt + \frac{1}{(1+x^2)^2}\int_0^x \frac{t^2}{(1+t^2)^{\frac{5}{2}}}\,dt$$

$$= \frac{(3x + 2x^3)}{3(1+x^2)^2} + \frac{1}{3}\frac{3x + 2x^3}{(1+x^2)^2}\left[\frac{1}{(1+x^2)^{\frac{3}{2}}} - 1\right] + \frac{1}{3(1+x^2)^2}\left[\frac{x^3}{(1+x^2)^{\frac{3}{2}}} - 0\right]$$

$$= \frac{3x + 2x^3}{3(1+x^2)^{\frac{7}{2}}} + \frac{x^3}{3(1+x^2)^{\frac{7}{2}}}$$

$$= \frac{x}{(1+x^2)^{\frac{5}{2}}}$$

$$= \varphi(x).$$

Thus, $\varphi(x) = \frac{x}{(1+x^2)^{\frac{5}{2}}}$ is a solution of the given integral equation (1.9).

1.7 DIFFERENTIATION OF A FUNCTION UNDER AN INTEGRAL SIGN:

Consider the function $I_n(x)$ defined by the relation

$$I_n(x) = \int_a^x (x - \eta)^{n-1} f(\eta) d\eta, \qquad (1.11)$$

where η is a positive integer and a is a constant.

We know that

$$\frac{d}{dx} \int_{p(x)}^{q(x)} G(x,\eta)\, d\eta = \int_{p(x)}^{q(x)} \partial/\partial x \{G(x,\eta)\}\, d\eta + G\big(x, q(x)\big) \frac{dq(x)}{dx} - G\big(x, p(x)\big) \frac{dp(x)}{dx},$$

which is valid if G and $\partial G/\partial x$ are continuous of both x, η and the first derivative of $p(x)$ and $q(x)$ are continuous.

Differentiating (1.11) under the integral sign, we have

$$\frac{dI_n}{dx} = (n-1) \int_a^x (x - \eta)^{n-2} f(\eta)\, d\eta + [(x - \eta)^{n-1} f(\eta)]_{\eta = x} \frac{d}{dx}(x)$$

$$- [(x - \eta)^{n-1} f(\eta)]_{\eta = a} \frac{d}{dx}(a)$$

$$\frac{dI_n}{dx} = (n-1) I_{n-1}, n > 1. \qquad (1.12)$$

From the relation (1.11), we have

$$I_1(x) = \int_a^x f(\eta)\, d\eta \Rightarrow \frac{dI_1}{dx} = f(x) \qquad (1.13)$$

Differentiating (1.12) successively m times, we have

$$\frac{d^m I_n}{dx^m} = (n-1)(n-2)\ldots(n-m) I_{n-m}, \qquad n > m.$$

In particular, we have

$$\frac{d^{n-1} I_n}{dx^{n-1}} = (n-1)!\; I_1(x)$$

$$\frac{d}{dx}\left[\frac{d^{n-1} I_n}{dx^{n-1}}\right] = (n-1)! \frac{dI_1}{dx} = (n-1)!\; f(x). \qquad (1.14)$$

Thus, we have

$$I_1(x) = \int_a^x f(x_1)dx_1$$

$$\frac{dI_2}{dx} = I_1 = \int_a^x f(x_1)dx_1$$

$$I_2(x) = \int_a^x \int_a^{x2} f(x_1)dx_1 dx_2.$$

In general, we have

$$I_n(x) = (n-1)! \int_a^x \int_a^{x_n} \dots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1\, dx_2 \dots dx_{n-1} dx_n\,. \qquad (1.15)$$

From the relations (1.11) and (1.15), we have

$$\int_a^x \int_a^{x_n} \dots \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1\, dx_2 \dots dx_{n-1} dx_n = \frac{1}{(n-1)!} I_n(x)$$

$$= \frac{1}{(n-1)!} \int_a^x (x - \eta)^{n-1} f(\eta)\, d\eta.$$

i.e.,

$$\int_a^x f(\eta) d\eta^n = \int_a^x \frac{(x - \eta)^{n-1}}{(n-1)!} f(\eta) d\eta.$$

1.8 RELATIONSHIP BETWEEN LINEAR DIFFERENTIAL EQUATIONS AND VOLTERRA INTEGRAL EQUATIONS:

The solution of the linear differential equation

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n(x)\, y = F(x) \qquad (1.16)$$

with continuous coefficients $a_i(x)$ $(i = 1, 2, \dots, n)$, given initial conditions

$$y(0) = C_0, y'(0) = C_1, \dots, y^{(n-1)}(0) = C_{n-1} \qquad (1.17)$$

may be reduced to a solution of a Volterra integral equation of the second kind.

Let us demonstrate this in the case of a differential equation of the second order.

Let

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x), \qquad (1.18)$$

$$y(0) = C_0, y'(0) = C_1. \qquad (1.19)$$

Put

$$\frac{d^2y}{dx^2} = \varphi(x). \qquad (1.20)$$

Integrating both sides of (1.20) from 0 to x, we have

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x \varphi(t)\,dt$$

$$\frac{dy}{dx} - y'(0) = \int_0^x \varphi(t)\,dt$$

$$\frac{dy}{dx} = C_1 + \int_0^x \varphi(t)\,dt. \qquad (1.21)$$

Integrating both sides of (1.21) from 0 to x, we have

$$y(x) - y(0) = C_1\,[t]_0^x + \int_0^x \int_0^x \varphi(t_1)dt_1\,dt$$

$$y(x) = C_0 + C_1\,x + \int_0^x \frac{(x - t)^1}{1!}\,\varphi(t)dt. \qquad (1.22)$$

Putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y given by $(1.20), (1.21)$ and (1.22) respectively in (1.18), we get

$$\varphi(x) + a_1(x)[C_1 + \int_0^x \varphi(t)dt] + a_2(x)\,[C_0 + C_1\,x + \int_0^x (x - t)\varphi(t)dt] = F(x)$$

$$\varphi(x) = F(x) - C_1a_1(x) - C_0a_2(x) - C_1xa_2(x) - \int_0^x [a_1(x) + a_2(x)(x - t)]\,\varphi(t)\,dt$$

(or)

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt, \qquad (1.23)$$

where $f(x) = F(x) - C_1\,a_1(x) - C_0\,a_2(x) - C_1\,x\,a_2\,(x), \qquad (1.24)$

$$\lambda = -1, \qquad (1.25)$$

and $K(x,t) = a_1(x) + a_2(x)(x - t), \qquad (1.26)$

which represents the Volterra integral integral equation of the second kind.

The existence of a unique solution of equation (1.23) follows from the existence and uniqueness of solution of the Cauchy problem $(1.18) - (1.19)$ for a linear differential equation with continuous coefficients in the neighborhood of the point $x = 0$.

Conversely, solving the integral equation (1.23) with f, λ and K determined from $(1.24), (1.25)$ and (1.26), and substituting the expression obtained for $\varphi(x)$ into the equation (1.22), we get a unique solution to equation (1.18) which satisfies the initial conditions (1.19).

Example 1.3 Form an integral equation corresponding to the differential equation

$$y'' + xy' + y = 0$$

with the initial conditions

$$y(0) = 1, y'(0) = 0.$$

Solution: The given differential equation is

$$y'' + xy' + y = 0 \qquad (1.20)$$

subject to the initial conditions:

$$y(0) = 1, y'(0) = 0. \qquad (1.21)$$

Suppose that

$$\frac{d^2y}{dx^2} = \varphi(x). \qquad (1.22)$$

Integrating both sides of (1.22) from 0 to x, we have

$$\left[\frac{dy}{dx}\right]_0^x = \int_0^x \varphi(t)\,dt$$

$$\frac{dy}{dx} - y'(0) = \int_0^x \varphi(t)\,dt$$

$$\frac{dy}{dx} = \int_0^x \varphi(t)\,dt. \qquad (1.23)$$

Integrating both sides of (1.23) from 0 to x, we have

$$[y(x) - y(0)] = \int_0^x \int_0^t \varphi(t_1)dt_1\,dt$$

$$y(x) = 1 + \int_0^x (x-t)\varphi(t)dt. \qquad (1.24)$$

Putting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y given by (1.22), (1.23) and (1.24) respectively in (1.20), we get

$$\varphi(x) + x\left[\int_0^x \varphi(t)dt\right] + 1 + \int_0^x (x-t)\varphi(t)dt = 0$$

$$\varphi(x) = -1 - \int_0^x [x + (x-t)]\varphi(t)dt$$

$$\varphi(x) = -1 - \int_0^x (2x-t)\varphi(t)dt$$

$$\varphi(x) = f(x) + \lambda\int_0^x K(x,t)\varphi(t)dt,$$

where $f(x) = -1$, $\lambda = -1, K(x,t) = 2x - t$, which represents the Volterra integral equation of the second kind.

Example 1.4: Form an integral equation corresponding to the differential equation

$$y''' - 3xy = 0$$

with the initial conditions

$$y(0) = \frac{1}{2}, y'(0) = y''(0) = 1.$$

Solution:

Given differential equation is:

$$y''' - 3xy = 0 \qquad (1.25)$$

subject to the initial conditions:

$$y(0) = 1/2, y'(0) = y''(0) = 1. \quad (1.26)$$

Suppose that:

$$\frac{d^3y}{dx^3} = \varphi(x). \qquad (1.27)$$

Integrating both sides of (1.27) from 0 to x, we have:

$$\left[\frac{d^2y}{dx^2}\right]_0^x = \int_0^x \varphi(t)\ dt$$

$$\frac{d^2y}{dx^2} - y''(0) = \int_0^x \varphi(t)\ dt$$

$$\frac{d^2y}{dx^2} = 1 + \int_0^x \varphi(t)\ dt. \qquad (1.28)$$

Integrating both sides of (1.28) from 0 to x, we have:

$$\frac{dy}{dx} = 1 + x + \int_0^x \int_0^t \varphi(t_1)\, dt_1 dt \qquad (1.29)$$

(or)

$$\frac{dy}{dx} = 1 + x + \int_0^x (x - t)\varphi(\mathrm{t})dt.$$

Integrating both sides of (1.29) from 0 to x, we have

$$y(x) - y(0) = [t]_0^x + \left[\frac{t^2}{2}\right]_0^x + \int_0^x \int_0^t \int_0^{t_1} \varphi(t_2)dt_2dt_1dt$$

$$y(x) = \frac{1}{2} + x + \frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2!}\varphi(t)dt. \quad (1.30)$$

Putting the values of $\frac{d^3y}{dx^3}$ and y given by (1.27) and (1.30) respectively in (1.25), we have

$$\varphi(x) - 3x\left[\frac{1}{2} + x + \frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2!}\varphi(t)dt\right] = 0$$

$$\varphi(x) = \frac{3x(x+1)^2}{2} + \frac{3}{2}\int_0^x x(x-t)^2\varphi(t)dt$$

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\,\varphi(t)dt,$$

where

$$f(x) = \frac{3}{2}x(x+1)^2, \lambda = \frac{3}{2}, K(x,t) = x(x-t)^2,$$

which represents the Volterra integral equation of the second kind.

1.9 SUMMARY:

This lesson provided the basic concepts of the integral equations, namely, linear, non-linear, homogeneous, non-homogeneous, and different kinds of integral equations. In this unit, we are mainly focusing on the Volterra integral equations. Next, we explain the relation between linear differential equations and Volterra integral equations. Finally, we have given examples and self-assessment problems that we included for better understanding of the readers.

1.10 TECHNICAL TERMS:

Integral equation, linear, non-linear, homogenous, non-homogeneous, Volterra integral equation, Fredholm integral equation.

1.11 SELF-ASSESSMENT QUESTIONS:

(1a) Verify that the given functions are solutions of the corresponding integral equations:

1. $\varphi(x) = e^x(cose^x - e^x\, sine^x)$;

   $$\varphi(x) = (1 - xe^{2x})cos\,1 - e^x\, sin\,1 + \int_0^x [1 - (x-t)e^{2x}]\varphi(t)dt.$$

2. $\varphi(x) = xe^x; \varphi(x) = e^x \sin x + 2\int_0^x cos(x-t)\,\varphi(t)\,dt.$

3. $\varphi(x) = x - \frac{x^3}{6};\ \varphi(x) = x - \int_0^x sinh(x-t)\,\varphi(t)\,dt.$

4. $\varphi(x) = 1 - x; \int_0^x e^{x-t}\,\varphi(t)\,dt = x.$

5. $\varphi(x) = 3; x^3 = \int_0^x (x-t)^2\,\varphi(t)\,dt.$

6. $\varphi(x) = 1 / (2\sqrt{x}); \int_0^x \varphi(t)/\sqrt{(x - t)}\, dt = \sqrt{x}.$

7. $\varphi(x) = 1 / (\pi\sqrt{x}); \int_0^x \varphi(t) / \sqrt{(x - t)}\, dt = 1.$

(1b) Form the integral equations corresponding to the following differential equations with the given initial conditions:

1. $y'' + y = 0;\ y(0) = 0, y'(0) = 1.$
2. $y' - y = 0;\ y(0) = 1.$
3. $y'' + y = cos\, x;\ y(0) = y'(0) = 0.$
4. $y'' - 5y' + 6y = 0;\ y(0) = 0, y'(0) = 1.$
5. $y'' + y = cos\, x;\ y(0) = 0, y'(0) = 1.$
6. $y'' - y' \sin x + e^x y = x;\ y(0) = 1, y'(0) = -1.$
7. $y'' + (1 + x^2)\, y = cos\, x;\ y(0) = 0, y'(0) = 2.$
8. $y''' + xy'' + (x^2 - x)\, y = xe^x + 1;\ y(0) = y'(0) = 1, y''(0) = 0.$
9. $y''' - 2xy = 0;\ y(0) = \frac{1}{2}, y'(0) = y''(0) = 1.$

Solutions to Self-Assessment Questions:

Exercise (1b):

1. $\varphi(x) = -x + \int_0^x (t - x)\varphi(t)dt$
2. $\varphi(x) = 1 + \int_0^x \varphi(t)dt$
3. $\varphi(x) = cos\, x - \int_0^x (x - t)\varphi(t)dt$
4. $\varphi(x) = 5 - 6\, x + \int_0^x [5 - 6(x - t)]\varphi(t)dt$
5. $\varphi(x) = cos\, x - x - \int_0^x (x - t)\varphi(t)dt$
6. $\varphi(x) = x - sin\, x + e^x(x - 1) + \int_0^x \left[sin\, x - e^x(x - t)\right]\varphi(t)dt$
7. $\varphi(x) = cos\, x - 2x(1 + x^2) - \int_0^x (1 + x^2)(x - t)\varphi(t)dt$
8. $\varphi(x) = xe^x + 1 - x(x^2 - 1) - \int_0^x \left[x + \frac{1}{2}(x^2 - x)(x - t)^2\right]\varphi(t)dt$
9. $\varphi(x) = x(x + 1)^2 + \int_0^x x(x - t)^2\varphi(t)dt$

1.12 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971 by M. Krasnov, A. Kiselev and G. Makarenko.
2. Integral equations, Krishna's Educational Publishers, Meerut- 250001, India, 1975 by Shanti Swarup and Shiv Raj Singh.
3. Integral Equations and Boundary Value Problems, S. Chand & Company PVT. LTD, New Delhi-110055, India, 2007 by Dr. M.D. Raisinghania.
4. Integral Equations and their Applications, WIT Press, 25 Bridge Street, Billerica, MA 01821, USA, by M. Rahman.
5. Introduction to Integral Equations with Applications, John Wiley & Sons, 1999, by Jerri, A.
6. Linear Integral Equation, Theory and Techniques, Academic Press, 2014, by Kanwal R. P.

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LESSON - 2

SOLUTION OF INTEGRAL EQUATION BY USING RESOLVENT KERNEL

OBJECTIVES:

- To determine the resolvent kernel by the method of iterated kernel
- To determine the resolvent kernel if the kernel is a polynomial in t
- To determine the resolvent kernel if the kernel is a polynomial in x
- To determine the resolvent kernel if the kernel takes the form $K(x - t)$

STRUCTURE:

2.1 Finding Resolvent Kernel using Iterated Kernels

2.2 Determination of Resolvent Kernel

Method 1: If $K(x,t)$ is a polynomial of degree $n-1$ in t

Method 2: If $K(x,t)$ is a polynomial of degree $n-1$ in x

Method 3: If $K(x,t)$ is of the form $K(x-t)$

2.3 Summary

2.4 Technical Terms

2.5 Self-Assessment Questions

2.6 Suggested Readings

2.1 FINDING RESOLVENT KERNEL USING ITERATED KERNELS:

Consider the Volterra integral equation of second kind:

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt, \qquad (2.1)$$

where the kernel $K(x,t)$ is a continuous function for $0 \le t \le x, 0 \le x \le a$ and the function $f(x)$ is continuous for $0 \le t \le x, 0 \le x \le a$.

Consider an infinite power series in ascending powers of λ as:

$$\varphi(x) = \varphi_0(x) + \lambda\,\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\,\varphi_n(x) + \cdots. \qquad (2.2)$$

Let the series (2.2) is a solution of the integral equation (1), then

$$\varphi_0(x) + \lambda\,\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots \lambda^n\,\varphi_n(x) + \cdots = f(x) + \lambda \int_0^x K(x,t)\,[\varphi_0(t) + \lambda\,\varphi_1(t) + \lambda^2\varphi_2(t) + \cdots \lambda^n\,\varphi_n(t) + \cdots]dt. \qquad (2.3)$$

Equating the coefficients of like powers of λ, we get

$$\varphi_0(x) = f(x)$$

$$\varphi_1(x) = \int_0^x K(x,t)\varphi_0(t)dt$$

$$\varphi_2(x) = \int_0^x K(x,t)\varphi_1(t)dt$$

$$\vdots$$

$$\varphi_n(x) = \int_0^x K(x,t)\varphi_{n-1}(t)dt. \qquad (2.4)$$

Thus, it yields a method for the successive approximation of the functions $\varphi_n(x)$. It may be shown that the series (2.2) converges uniformly in x and λ for any λ and $x \in [0,a]$, under these assumptions with regard to $f(x)$ and $K(x,t)$, its sum is a unique solution of the equation (2.1). Further, from (2.4), it follows that

$$\varphi_1(x) = \int_0^x K(x,t)f(t)dt,$$

$$\varphi_2(x) = \int_0^x K(x,t)\left\{\int_0^t K(t,t_1)f(t_1)dt_1\right\}dt.$$

Here, $t_1 = 0, t_1 = t; t = 0, t = x$.

By interchanging the order of integration, we have

$$\varphi_2(x) = \int_0^x f(t_1)dt_1\left\{\int_{t_1}^x K(x,t)K(t,t_1)\,dt\right\}$$

$$\varphi_2(x) = \int_0^x K_2(x,t_1)f(t_1)dt_1, \qquad (2.5)$$

where

$$K_2(x,t_1) = \int_{t_1}^x K(x,t)K(t,t_1)\,dt. \qquad (2.6)$$

In general, we have

$$\varphi_n(x) = \int_0^x K_n(x,t)f(t)dt, \quad n = 1,2,3,\cdots. \qquad (2.7)$$

The functions $K_n(x,t)$ are called iterated kernels, which can readily be shown that

$$K_1(x,t) = K(x,t)$$

and $K_2(x,t), K_3(x,t)$ etc., are defined recursively by the formulas

$$K_{n+1}(x,t) = \int_t^x K(x,z)K_n(z,t)\,dz, n = 1,2,3,\cdots. \qquad (2.8)$$

The relation (2.2), which represents the solution of the integral equation (2.1) can therefore be written as

$$\varphi(x) = f(x) + \sum_{v=1}^{\infty}\lambda^{v-1}\int_0^x K_v(x,t)f(t)dt \qquad (2.9)$$

$$\varphi(x) = f(x) + \int_0^x \sum_{v=1}^{\infty}\lambda^{v-1}K_v(x,t)f(t)dt$$

$$\varphi(x) = f(x) + \int_0^x R(x,t;\lambda) f(t)dt, \qquad (2.10)$$

where

$$R(x,t;\lambda) = \sum_{v=1}^{\infty} \lambda^{v-1} K_v(x,t). \qquad (2.11)$$

The function $R(x,t;\lambda)$ is called the resolvent kernel or reciprocal kernel of the integral equation (2.1). Series (2.11) converges absolutely and uniformly in the case of a continuous kernel $K(x,t)$.

Iterated kernels and also the resolvent kernel do not depend on the lower limit in an integral equation.

The resolvent kernel $R(x,t;\lambda)$ satisfies the following functional equation:

$$R(x,t;\lambda) = K(x,t) + \lambda \int_t^x K(x,s) R(s,t;\lambda) ds. \qquad (2.12)$$

With the aid of the resolvent kernel, the solution of the integral equation (2.1) may be written in the form,

$$\varphi(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t)dt. \qquad (2.13)$$

Example 2.1:

Find the resolvent kernel of the Volterra integral equation with kernel $k(x,t) \equiv 1$.

Solution.

We know that the iterated kernels $K_n(x,t)$ are given by:

$$K_1(x,t) = K(x,t), \qquad (2.14)$$

$$K_n(x,t) = \int_t^x K(x,z) K_{n-1}(z,t) dz, \quad n = 2,3,\cdots. \qquad (2.15)$$

Given

$$K(x,t) = 1. \qquad (2.16)$$

From (2.14) and (2.16),

$$K_1(x,t) = K(x,t) = 1. \qquad (2.17)$$

Putting $n = 2$ in (2.15) and using (2.17), we have:

$$K_2(x,t) = \int_t^x K(x,z) K_1(z,t)\, dz = \int_t^x 1 \cdot 1 \cdot dz = [z]_t^x = x - t. \qquad (2.18)$$

Next, putting $n = 3$ in (2.15), we have:

$$K_3(x,t) = \int_t^x K(x,z) K_2(z,t) dz = \int_t^x 1 \cdot (z - t) dz \quad \text{(using (2.17) and (2.18))}$$

$$= \left[\frac{(z-t)^2}{2}\right]_t^x = \frac{(x-t)^2}{2!}. \qquad (2.19)$$

And so, putting $n = 4$ in (2.15), we have:

$$K_4(x,t) = \int_t^x K(x,z)K_3(z,t)dz = \int_t^x 1 \cdot \frac{(z-t)^2}{2!}\, dz \quad \text{(using (2.17) and (2.19))}$$

$$= \frac{1}{2!}\left[\frac{(z-t)^3}{3}\right]_t^x = \frac{(x-t)^3}{3!}$$

$$\vdots$$

$$K_n(x,t) = \int_t^x K(x,z)K_{n-1}(z,t)dz = \frac{(x-t)^{n-1}}{(n-1)!}, \qquad for\, n = 1,2,3,\cdots.$$

Thus, by definition of the resolvent kernel

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1}\, K_n(x,t)$$

$$R(x,t;\, \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1}\, \frac{(x-t)^{n-1}}{(n-1)!}.$$

Therefore, the resolvent kernel is $R(x,t;\, \lambda) = e^{\lambda(x-t)}$.

Example 2.2:

Find the resolvent kernel of the Volterra integral equation with kernel $K(x,t) = x - t$.

Solution.

We know that the iterated kernels $K_n(x,t)$ are given by:

$$K_1(x,t) = K(x,t), \qquad (2.20)$$

$$K_n(x,t) = \int_t^x K(x,z)K_{n-1}(z,t)dz, \quad n = 2,3,\cdots. \qquad (2.21)$$

Given

$$K(x,t) = x - t. \qquad (2.22)$$

From (2.20) and (2.22),

$$K_1(x,t) = K(x,t) = x - t. \qquad (2.23)$$

Putting $n = 2$ in (2.21) and using (2.23), we have:

$$K_2(x,t) = \int_t^x k(x,z)K_1(z,t)\, dz$$

$$= \int_t^x (x-z)(z-t)dz$$

$$= \int_t^x [-z^2 + (x+t)z - xt]\, dz$$

$$= \frac{1}{6}(x^3 - 3x^2 t + 3xt^2 - t^3).$$

Thus,

$$K_2(x,t) = \frac{(x-t)^3}{3!}. \tag{2.24}$$

Next, putting $n = 3$ in (2.21), we have:

$$K_3(x,t) = \int_t^x k(x,z) K_2(z,t) dz$$

$$= \int_t^x (x-z) \cdot \frac{(z-t)^3}{3!}\, dz \quad \big(\text{using } (2.22) \text{and } (2.24)\big)$$

$$= \frac{(x-t)^5}{5!}. \tag{2.25}$$

Next, putting $n = 4$ in (2.21), we have:

$$K_4(x,t) = \int_t^x k(x,z) K_3(z,t) dz$$

$$= \int_t^x (x-z) \cdot \frac{(z-t)^5}{5!}\, dz \quad \big(\text{using } (2.22) \text{and } (2.25)\big)$$

$$= \frac{(x-t)^7}{7!}$$

$$\vdots$$

$$K_n(x,t) = \int_t^x k(x,z) K_{n-1}(z,t) dz = \frac{(x-t)^{2n-1}}{(2n-1)!}, for\ n = 1, 2, 3, \cdots.$$

Thus, by definition of the resolvent kernel

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K_n(x,t)$$

$$R(x,t;\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} \frac{(x-t)^{2n-1}}{(2n-1)!}.$$

Therefore, the resolvent kernel is $R(x,t;\lambda) = \frac{1}{\sqrt{\lambda}} \sinh\big[\sqrt{\lambda}(x-t)\big], \lambda > 0.$

2.2 DETERMINATION OF RESOLVENT KERNEL:

Method (1):

Let the kernel $K(x,t)$ be in the form of a polynomial of degree $(n-1)$ in t such that it may be represented in the form:

$$K(x,t) = a_0(x) + a_1(x)(x-t) + \dots + \frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1} \qquad (2.26)$$

where the coefficients $a_k(x)$ are continuous in $[0,a]$, $k = 0,1,\cdots,n-1$.

Let the auxiliary function be:

$$g(x,t;\lambda) = \frac{1}{(n-1)!}(x-t)^{n-1} + \lambda\int_t^x R(z,t;\lambda)\,\frac{(x-z)^{n-1}}{(n-1)!}\,dz \qquad (2.27)$$

with the conditions:

$$g|_{x=t} = \frac{dg}{dx}|_{x=t} = \ldots = \frac{d^{n-2}g}{dx^{n-2}}|_{x=t} = 0 \text{ and } \frac{d^{n-1}g}{dx^{n-1}}|_{x=t} = 1. \quad (2.28)$$

In addition, we have:

$$R(x,t;\lambda) = \frac{1}{\lambda}\frac{d^n g(x,t;\lambda)}{dx^n}. \qquad (2.29)$$

Since the resolvent kernel satisfies the functional equation:

$$R(x,t;\lambda) = K(x,t) + \lambda\int_t^x K(x,z)\frac{d^n g(z,t;\lambda)}{dz^n}\,dz. \qquad (2.30)$$

From (2.29) and (2.30), we have:

$$\frac{d^n g(x,t;\lambda)}{dx^n} = \lambda\,K(x,t) + \lambda\int_t^x K(x,z)\,\frac{d^n g(z,t;\lambda)}{dz^n}\,dz \qquad (2.31)$$

$$\frac{d^n}{dx^n}g(x,t;\lambda) =$$
$$\lambda\,K(x,t) +$$
$$\lambda\left[K(x,z)\frac{d^{n-1}g}{dz^{n-1}} - \frac{\partial K(x,z)}{\partial z}\frac{d^{n-2}g}{dz^{n-2}} + \cdots + \frac{\partial^{n-1}K(x,z)}{\partial z^{n-1}}\right]_{z=t}^{z=x} \qquad (2.32)$$

using (2.26) and (2.28), the relation (2.32) reduces to

$$\frac{d^n g}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}g}{dx^{n-1}} + a_1(x)\frac{d^{n-2}g}{dx^{n-2}} + \ldots + a_{n-1}(x)\,g\right]_{z=t}^{z=x}. \quad (2.33)$$

The function $g(x,t;\lambda)$ is therefore the integral of the solution of the differential equation (2.33).

Thus, we have an expression for the resolvent kernel as:

$$R(x,t;\lambda) = \frac{1}{\lambda}\frac{d^n}{dx^n}g(x,t;\lambda).$$

Method (2):

Assume that the kernel $K(x,t)$ is a polynomial of degree $(n-1)$ in x such that it may be represented in the form:

$$K(x,t) = b_0(t) + b_1(t)(t-x) + \ldots + \frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1} \qquad (2.34)$$

where the coefficients $b_v(t)$ are continuous in $[0,a], v = 0,1,\cdots,n-1$.

Consider,

$$R(x,t;\lambda) = -\frac{1}{\lambda}\frac{d^n g(t,x;\lambda)}{dt^n}. \qquad (2.35)$$

The auxiliary function $g(t,x;\lambda)$ satisfies the following conditions:

$$g|_{t=x} = \frac{dg}{dt}|_{t=x} = \cdots = \frac{d^{n-2}g}{dt^{n-2}}|_{t=x} = 0 \text{ and } \frac{d^{n-1}g}{dt^{n-1}}|_{t=x} = 1. \qquad (2.36)$$

Therefore, the functional relation reduces to

$$\frac{d^n g}{dt^n} = \lambda\, K(x,t) + \lambda \int_t^x K(x,z)\frac{d^n}{dz^n}g(t,z;\lambda)dz. \qquad (2.37)$$

Using the expression (2.35) and (2.36) and integrating by parts to the integral on R.H.S., we have

$$\frac{d^n g}{dx^n} + \lambda\left[b_0(t)\frac{d^{n-1}g}{dt^{n-1}} + b_1(t)\frac{d^{n-2}g}{dt^{n-2}} + \ldots + b_{n-1}(t)\, g\right]. \qquad (2.38)$$

The function $g(t,x;\lambda)$ is therefore the integral of the solution of the differential equation (2.38).

Hence, the resolvent of the kernel is

$$R(x,t;\lambda) = -\frac{1}{\lambda}\frac{d^n}{dt^n}g(t,x;\lambda). \qquad (2.39)$$

Example. 2.3:

Find the resolvent kernel for the integral equation with the following kernel

$(\lambda = 1)$

$$K(x,t) = 2-(x-t).$$

Solution. Here $K(x,t) = 2-(x-t), \lambda = 1$.

Comparing with the relation

$$K(x,t) = a_0(x) + a_1(x)(x-t) + \cdots + \frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1},$$

we have $a_0(x) = 2,\ a_1(x) = -1$, and all the other $a_v(x) = 0$.

Thus, the equation

$$\frac{d^n g}{dx^n} - \lambda\left[a_0(x)\frac{d^{n-1}g}{dx^{n-1}} + a_1(x)\frac{d^{n-2}g}{dx^{n-2}} + \cdots + a_{n-1}(x)g\right] = 0$$

reduces to

$$\frac{d^2g}{dx^2} - 2\frac{dg}{dx} + g = 0, \tag{2.40}$$

with the condition $g = 0$ at $x = t, \frac{dg}{dx} = 1$ at $x = t$. (2.41)

The solution of equation (2.40) is given by

$$g = [A(t) + B(t)x]e^t. \tag{2.42}$$

From (2.41) and (2.42), we obtain

$$g = g(x, t; 1) = (x - t)e^{x-t}.$$

Thus, the resolvent kernel is given by

$$R(x, t;\ 1) = \frac{1}{\lambda}\frac{d^2}{dx^2}\ g(x, t;\ 1) = (x - t + 2)e^{x-t}.$$

Example 2.4:

Find the resolvent kernel for the integral equation

$$\varphi(x) = (\cos x - x - 2) + \int_0^x (t - x)\varphi(t)dt.$$

Solution. Here $f(x) = cos x - x - 2, \lambda = 1\ and\ K(x, t) = t - x$.

Comparing with the relation

$$K(x, t) = b_0(t) + b_1(t)(t - x) + \cdots + \frac{b_{n-1}(t)}{(n-1)!}(t - x)^{n-1}$$

we have $b_1(t) = 1$, and all the other $b_v(t) = 0$.

Thus, the equation

$$\frac{d^n g}{dt^n} + \lambda\left[b_0(t)\frac{d^{n-1}g}{dt^{n-1}} + b_1(t)\frac{d^{(n-2)}g}{dt^{n-2}} + \cdots + b_{n-1}(t)g\right] = 0$$

reduces to

$$\frac{d^2g}{dt^2} + g = 0 \tag{2.43}$$

with the conditions

$$g = 0 \text{ at } t = x \text{ and } \frac{dg}{dt} = 1 \text{ at } t = x. \tag{2.44}$$

The solution of the equation (2.43) is given by

$$g(t, x;\ 1) = A(x)cos\ t + B(x)sin\ t. \tag{2.45}$$

From (2.44) and (2.45), we obtain

$$g(t, x; 1) = sin(t - x).$$

Hence, the resolvent kernel becomes

$$R(x,t;1) = -\frac{1}{\lambda}\frac{d^2}{dt^2}g(t,x;1) = sin(t-x).$$

Method (3):

Result: Suppose we have a Volterra-type integral equation, the kernel of which is dependent solely on the difference of the arguments.

$$\varphi(x) = f(x) + \int_0^x K(x-t)\varphi(t)dt\,, (\lambda = 1). \qquad (2.45)$$

Show that for the equation (2.44) all iterated kernels and the resolvent kernel are also dependent solely on the difference $x - t$.

Proof: Let the functions $f(x)$ and $K(x)$ in (2.45) be original functions. Taking the Laplace transform of both sides of (2.45) and employing the product theorem (transform of a convolution), we get

$$\Phi(p) = F(p) + \widetilde{K}(p)\Phi(p),$$

where

$$L\{\varphi(x)\} = \Phi(p),$$
$$L\{f(x)\} = F(p),$$
$$L\{K(x)\} = \widetilde{K}(p).$$

Hence, $$\Phi(p) = \frac{F(p)}{1 - \widetilde{K}(p)},\quad \widetilde{K}(p) \neq 1. \qquad (2.46)$$

We can write the solution of the integral equation (2.45) in the form

$$\varphi(x) = f(x) + \int_0^x R(x-t)f(t)dt, \qquad (2.47)$$

where $R(x-t)$ is the resolvent kernel for the integral equation (2.45).

Taking the Laplace transform of both sides of the equation (2.47)

$$\Phi(p) = F(p) + \tilde{R}(p)F(p),$$

where $L\{R(x)\} = \tilde{R}(p)$.

Hence

$$\tilde{R}(p) = \frac{\Phi(p) - F(p)}{F(p)}. \qquad (2.48)$$

Substituting into (2.47) the expression for $\Phi(p)$ from (2.45), we obtain

$$\tilde{R}(p) = \frac{\widetilde{K}(p)}{1 - \widetilde{K}(p)}. \qquad (2.49)$$

Apply the inverse Laplace on both sides, we get $R(x)$.

Resolvent kernel for the integral equation (2.45) is

$$R(x,t;\ 1) = R(x-t).$$

Example 2.5:

Find the resolvent kernel for a Volterra integral equation $K(x,t) = sin(x-t),\ \lambda = 1$.

Solution: Given that $K(x,t) = sin(x-t), \lambda = 1$. Then, $K(x) = sin(x)$.

Apply the Laplace transform on both sides, we get

$$L\{K(x)\} = L\{sin(x)\}$$

$$\widetilde{K}(p) = \frac{1}{1+p^2}.$$

Since

$$\tilde{R}(p) = \frac{\widetilde{K}(p)}{1-\widetilde{K}(p)}$$

$$\tilde{R}(p) = \frac{\left(\frac{1}{1+p^2}\right)}{1-\left(\frac{1}{1+p^2}\right)}$$

$$= \frac{1}{p^2}$$

$$\therefore \tilde{R}(p) = \frac{1}{p^2}.$$

Apply the Laplace inverse transform on both sides, we get

$$L^{-1}\{\tilde{R}(p)\} = L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$R(x) = x$$

Resolvent kernel $R(x,t;\ 1) = R(x-t) = x-t$.

Example 2.6:

Find the resolvent kernel for the Volterra integral equation $K(x,t) = e^{-(x-t)},\ \lambda = 1$.

Solution: Given that $K(x,t) = e^{-(x-t)}, \lambda = 1$, then

$$K(x) = e^{-x}.$$

Apply the Laplace transform on both sides, we get

$$L\{K(x)\} = L\{e^{-x}\}$$

$$\widetilde{K}(p) = \frac{1}{p+1}.$$

Since

$$\tilde{R}(p) = \frac{\widetilde{K}(p)}{1 - \widetilde{K}(p)} = \frac{\frac{1}{p+1}}{1 - \left(\frac{1}{p+1}\right)} = \frac{1}{p}$$

$$\therefore \tilde{R}(p) = \frac{1}{p}.$$

Apply the Laplace inverse transform on both sides, we get

$$L^{-1}\{\tilde{R}(p)\} = L^{-1}\left\{\frac{1}{p}\right\}$$

$$R(x) = 1.$$

$\therefore$ Resolvent kernel $R(x, t;\ 1) = R(x - t) = 1.$

Example 2.7:

With the aid of the resolvent kernel, find the solution of the integral equation

$$\varphi(x) = e^{x^2} + \int_0^x e^{x^2 - t^2} \varphi(t) dt.$$

Solution. Given that

$$\varphi(x) = e^{x^2} + \int_0^x e^{x^2 - t^2} \varphi(t) dt, \tag{2.50}$$

where $f(x) = e^{x^2}$, $K(x, t) = e^{x^2 - t^2}, \lambda = 1.$

Iterated kernels $K_n(x, t)$ are given by

$$K_1(x, t) = K(x, t) \tag{2.51}$$

and

$$K_n(x, t) = \int_t^x K(x, z) K_{n-1}(z, t) dz\ ,\quad n = 2, 3, \cdots. \tag{2.52}$$

Given $K(x, t) = e^{x^2 - t^2}.$

Putting $n = 2$ in (2.52), we have

$$\begin{aligned} K_2(x, t) &= \int_t^x K(x, z)\, K_1(z, t)\, dz \\ &= \int_t^x e^{x^2 - z^2} e^{z^2 - t^2} dz \\ &= e^{x^2 - t^2} \int_t^x 1\, dz \\ &= e^{x^2 - t^2} \frac{(x - t)^1}{1!}. \end{aligned}$$

Next, putting $n = 3$ in (2.52), we have

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz$$

$$= e^{x^2 - t^2} \int_t^x \frac{(z - t)^1}{1!}\, dz$$

$$= e^{x^2 - t^2} \frac{(x - t)^2}{2!}.$$

Similarly,

$$K_n(x,t) = \int_t^x K(x,z) K_{n-1}(z,t)\, dz$$

$$= \int_t^x e^{x^2 - z^2}\, e^{z^2 - t^2} \frac{(z - t)^{n-2}}{(n-2)!}\, dz$$

$$K_n(x,t) = e^{x^2 - t^2} \frac{(x - t)^{n-1}}{(n-1)!}.$$

Thus, by the definition of the resolvent kernel

$$R(x,t;\, \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1}\, K_n(x,t)$$

$$R(x,t;\, 1) = \sum_{n=1}^{\infty} (1)^{n-1}\, e^{x^2 - t^2} \frac{(x - t)^{n-1}}{(n-1)!}$$

$$= e^{x^2 - t^2} \sum_{n=1}^{\infty} (1)^{n-1} \frac{(x - t)^{n-1}}{(n-1)!}$$

$$R(x,t;\, 1) = e^{x^2 - t^2}\, e^{x - t}.$$

The solution of the integral equation (2.50) is

$$\varphi(x) = f(x) + \lambda \int_0^x R(x,t;\, \lambda)\, f(t)\, dt$$

$$\varphi(x) = e^{x^2} + \int_0^x e^{x^2 - t^2}\, e^{x - t}\, e^{t^2}\, dt$$

$$= e^{x^2} + e^{x^2 + x} \int_0^x e^{-t}\, dt$$

$$= e^{x^2} + e^{x^2 + x} [-e^{-t}]_0^x = e^{x^2 + x}.$$

Note 1. The unique solvability of Volterra-type integral equations of the second kind

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t) dt \qquad (2.53)$$

holds under considerably more general assumptions with respect to the function $f(x)$ and the kernel $K(x,t)$ than their continuity.

The L^2 space is a special case of an L^p space, which is also known as the Lebesgue space

Definition: Let X be a measure space. Given a complex function f, we say $f \in L^2$ on X if f is (Lebesgue) measurable and if

$$\int_X |f|^2 d\mu < +\infty.$$

Then the function f is also said to be square–integrable. In other words, L^2 is the set of square–integrable functions.

For

$$f \in \left(\int_X |f|^2 d\mu\right)^{\frac{1}{2}}.$$

We call $\|f\|$ the $L^2(\mu)$ norm of f.

Theorem: The Volterra integral equation of the second kind (2.53) whose kernel $K(x,t)$ and function $f(x)$ belong, respectively, to spaces $L_2\ (\Omega_0)\ and\ L_2(0,a)$, has one and only one solution in the space $L_2(0,a)$.

This solution is given by the formula

$$\varphi(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t) dt \tag{2.54}$$

where the resolvent kernel $R(x,t;\lambda)$ is determined by means of the series

$$R(x,t;\lambda) = \sum_{v=0}^{\infty} \lambda^v K_{v+1}(x,t) \tag{2.55}$$

which is made up of iterated kernels and converges almost everywhere.

Note 2. In questions of uniqueness of solution of an integral equation, an essential role is played by the class of functions in which the solution is sought (the class of summable, quadratically summable, continuous, etc., functions).

Thus, if the kernel $K(x,t)$ of a Volterra equation is bounded when x varies in some finite interval (a,b) so that $|K(x,t)| \leq M,\ M =$const, $x \in\ (a,b)$ and the constant term of $f(x)$ is summable in the interval (a,b), then the Volterra equation has, for any value of λ, a unique summable solution $\varphi(x)$ in the interval (a,b).

However, if we give up the requirement of summability of the solution, then the uniqueness theorem ceases to hold in the sense that the equation can have nonsummable solutions along with summable solutions.

P. S. Uryson constructed elegant examples of integral equations (see Examples 1 and 2 below) which have summable and nonsummable solutions even when the kernel $K(x,t)$ and the function $f(x)$ are continuous.

For simplicity, we consider $f(x) \equiv 0$ and examine the integral equation

$$\varphi(x) = \int_0^x K(x,t)\varphi(t)\,dt \tag{2.56}$$

where $K(x,t)$ is a continuous function.

The only summable solution of the equation (2.56) is $\varphi(x) \equiv 0$.

Example 2.8:

Let

$$K(x,t) = \begin{cases} te^{\frac{1}{x^2}-1}, & 0 \le t \le xe^{1-\frac{1}{x^2}}, \\ x, & xe^{1-\frac{1}{x^2}} \le t \le x, \\ 0, & t > x. \end{cases} \tag{2.57}$$

The kernel $K(x,t)$ is bounded in the square $\Omega_0\,\{0 \le x, t \le 1\}$, since $0 \le K(x,t) \le x \le 1$. What is more, it is continuous for $0 \le t \le x$. In this case, the equation (2.56) has an obviously summable solution $\varphi(x) \equiv 0$ and by virtue of what has been said, this equation does not have any other summable solutions.

On the other hand, direct verification convinces us that equation (2.56) has an infinity of nonsummable solutions in $(0,1)$ in the form

$$\varphi(x) = \frac{C}{x},$$

where C is an arbitrary constant and $x \neq 0$.

Indeed, taking into account expression (2.57) for the kernel $K(x,t)$, we find

$$\begin{aligned} \int_0^x K(x,t)\varphi(t)\,dt &= \int_0^{xe^{1-\frac{1}{x^2}}} te^{\frac{1}{x^2}-1}\frac{C}{t}\,dt + \int_{xe^{1-\frac{1}{x^2}}}^x x\frac{C}{t}\,dt \\ &= Cx + Cx\ln e^{\frac{1}{x^2}-1} \\ &= \frac{C}{x}. \end{aligned}$$

Thus, we obtain

$$\frac{C}{x} \equiv \frac{C}{x} \quad (x \neq 0).$$

This means that $\varphi(x) = \frac{C}{x}$ is a nonsummable solution of equation (2.56).

Example 2.9:

Let $0 \le t \le x < a$ ($a > 0$, in particular $a = +\infty$),

$$K(x,t) = \frac{2}{\pi}\frac{xt^2}{(x^6 + t^2)}. \tag{2.58}$$

The function $K(x,t)$ is even holomorphic everywhere, except at the point $(0,0)$. However, equation (2.56) with kernel (2.58) admits nonsummable solutions. Indeed, the equation

$$\psi(x) = \frac{2}{\pi}\int_0^x \frac{xt^2}{(x^6 + t^2)}\,\psi(t)dt - \frac{2}{\pi}\frac{\arctan x^2}{x^2} \tag{2.59}$$

has a summable solution since the function

$$f(x) = -\frac{2}{\pi}\frac{\arctan x^2}{x^2}$$

is bounded and continuous everywhere except at the point $x = 0$.

The function

$$\varphi(x) = \begin{cases} 0, & x = 0, \\ \psi(x) + \dfrac{1}{x^2}, & x > 0, \end{cases} \tag{2.60}$$

where $\psi(x)$ is a solution of (2.59) will now be a nonsummable solution of (2.56) with kernel (2.58).

Indeed, for $x > 0$ we have

$$\int_0^x K(x,t)\varphi(t)dt = \frac{2}{\pi}\int_0^x \frac{xt^2}{(x^6 + t^2)}\psi(t)\,dt + \frac{2}{\pi}\int_0^x \frac{x}{x^6 + t^2}\,dt. \tag{2.61}$$

By virtue of equation (2.57), the first term on the right of (2.59) is

$$\psi(x) + \frac{2}{\pi}\frac{arctan x^2}{x^2}$$

The second term yields

$$\frac{2}{\pi}\int_0^x \frac{xdt}{x^6 + t^2} = \frac{2}{\pi}\left(\frac{1}{x^2}\arctan\frac{t}{x^3}\right)\Big|_{t=0}^{t=x} = \frac{2}{\pi x^2}\arctan\frac{1}{x^2} \quad (x > 0).$$

Thus,

$$\int_0^x K(x,t)\,\varphi(t)\,dt = \psi(x) + \frac{2}{\pi}\frac{arctan\, x^2}{x^2} + \frac{2}{\pi x^2}arctan\frac{1}{x^2} = \psi(x) + \frac{1}{x^2} = \varphi(x),$$

which means that the function $\varphi(x)$ defined by (2.60) is a nonsummable solution of equation with kernel (2.58).

Example 2.10:

The equation

$$\varphi(x) = \int_0^x t^{x-t}\, \varphi(t)\, dt \quad (0 \le x, t \le 1)$$

has a unique continuous solution $\varphi(x) \equiv 0$. By direct substitution, we see that this equation also has an infinity of discontinuous solutions of the form

$$\varphi(x) = Cx^{x-1},$$

where C is an arbitrary constant.

2.3 SUMMARY:

In this lesson, we find the resolvent kernel by using the iterated kernels and different types of methods. Finally, we have given examples and self-assessment problems that we included for better understanding of the readers.

2.4 TECHNICAL TERMS:

Integral equation, kernel, resolvent kernel, iterated kernel.

2.5 SELF-ASSESSMENT QUESTIONS:

(2a): Find the resolvents for the Volterra-type integral equations with the following kernels:

1. $K(x,t) = x - t$.
2. $K(x,t) = e^{x-t}$.
3. $K(x,t) = e^{x^2 - t^2}$.
4. $K(x,t) = \frac{1 + x^2}{1 + t^2}$.
5. $K(x,t) = \frac{2 + \cos x}{2 + \cos t}$.
6. $K(x,t) = \frac{\cosh x}{\cosh t}$.
7. $K(x,t) = a^{x-t}$ $(a > 0)$.

(2b): Find the resolvent kernels of integral equations with the following kernels $(\lambda = 1)$:

1. $K(x,t) = 2 - (x - t)$.
2. $K(x,t) = -2 + 3(x - t)$.
3. $K(x,t) = 2x$.
4. $K(x,t) = -\frac{4x - 2}{2x + 1} + \frac{8(x - t)}{2x + 1}$.

(2c): Find the resolvent kernels for Volterra-type integral equations with the kernels ($\lambda = 1$):

1. $K(x,t) = sinh(x-t)$.
2. $K(x,t) = e^{-(x-t)}$.
3. $K(x,t) = e^{-(x-t)} sin(x-t)$.
4. $K(x,t) = cosh(x-t)$.
5. $K(x,t) = 2\,cos(x-t)$.

(2d): Using the results of the preceding examples, find (by means of resolvent kernels) solutions of the following integral equations:

1. $\varphi(x) = e^x + \int_0^x e^{x-t} \varphi(t)\, dt$.
2. $\varphi(x) = sin\, x + 2 \int_0^x e^{x-t} \varphi(t)\, dt$.
3. $\varphi(x) = x\, 3^x - \int_0^x 3^{x-t} \varphi(t)\, dt$.
4. $\varphi(x) = e^x sin\, x + \int_0^x \frac{2 + cos\, x}{2 + cos\, t} \varphi(t)\, dt$.
5. $\varphi(x) = 1 - 2x - \int_0^x e^{x^2 - t^2} \varphi(t)\, dt$.
6. $\varphi(x) = e^{x^2 + 2x} + 2 \int_0^x e^{x^2 - t^2} \varphi(t)\, dt$.
7. $\varphi(x) = 1 + x^2 + \int_0^x \frac{1 + x^2}{1 + t^2} \varphi(t)\, dt$.
8. $\varphi(x) = \frac{1}{1 + x^2} + \int_0^x sin(x-t)\, \varphi(t)\, dt$.
9. $\varphi(x) = x\, e^{\frac{x^2}{2}} + \int_0^x e^{-(x-t)} \varphi(t)\, dt$.
10. $\varphi(x) = e^{-x} + \int_0^x e^{-(x-t)} sin(x-t)\, \varphi(t)\, dt$.

Solutions to Self-Assessment Questions:

Exercise (2a)

1. $\frac{1}{\sqrt{\lambda}} sinh\sqrt{\lambda}(x-t)\ (\lambda > 0)$
2. $e^{(1+\lambda)(x-t)}$
3. $e^{\lambda(x-t)} e^{x2\,-t2}$
4. $\frac{1+x^2}{1+t^2} e^{\lambda(x-t)}$
5. $\frac{2+cos\, x}{2+cos\, t}\, e^{\lambda(x-t)}$

6. $\frac{cosh\, x}{cosht}\ e^{\lambda(x-t)}$
7. $a^{x-t}e^{\lambda(x-t)}$

Exercise (2b)

1. $e^{x-t}(x-t+2)$
2. $\frac{1}{4}e^{x-t}-\frac{9}{4}e^{-3(x-t)}$
3. $2xe^{x^2-t^2}$
4. $\frac{(4t^2+1)}{2(2t+1)^2}\left[\frac{8}{(4t2+1)}-4e^{(-2)}(x-t)\right]$

Exercise (2c)

1. $\frac{1}{\sqrt{2}}sinh\, \sqrt{2}(x-t)$
2. 1
3. $(x-t)e^{-(x-t)}$
4. $e^{\frac{x-t}{2}}\left[cosh\ \frac{\sqrt{5}}{2}\ (x-t)+\frac{1}{\sqrt{5}}sinh\frac{\sqrt{5}}{2}(x-t)\right]$
5. $2e^{x-t}(1+x-t)$

Exercise (2d)

1. $\varphi(x)=e^{2x}$
2. $\varphi(x)=\frac{1}{5}e^{3x}-\frac{1}{5}cos\, x+\frac{2}{5}sin\, x$
3. $\varphi(x)=3x(1-e^{-x})$
4. $\varphi(x)=e^{x}sin\, x+\left(2+cos\, x\right)e^{x}\ln\frac{3}{2+cos\, x}$
5. $\varphi(x)=e^{x^2-x}-2x$
6. $\varphi(x)=e^{x^2+2x}(1+2x)$
7. $\varphi(x)=e^{x}(1+x^2)$
8. $\varphi(x)=\frac{1}{1+x^2}+x\, arctan\, x-\frac{1}{2}\ln(1+x^2)$
9. $\varphi(x)=e^{\frac{x^2}{2}}(x+1)-1$
10. $\varphi(x)=e^{-x}\left(\frac{x^2}{2}+1\right)$

2.6 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971, by M. Krasnov, A. Kiselev, and G. Makarenko.
2. Integral equations, Krishna's Educational Publishers, Meerut- 250001, India, 1975, by Shanti Swarup and Shiv Raj Singh.
3. Integral Equations and Boundary Value Problems, S. Chand & Company PVT. LTD, New Delhi-110055, India, 2007, by Dr. M.D. Raisinghania.
4. Integral Equations and their Applications, WIT Press, 25 Bridge Street, Billerica, MA 01821, USA, by M. Rahman.
5. Introduction to Integral Equations with Applications, John Wiley & Sons, 1999, by Jerri, A.
6. Linear Integral Equation, Theory and Techniques, Academic Press, 2014, by Kanwal R. P.

- Prof. K. Rajendra Prasad

LESSON- 3

SOLUTION OF VOLTERRA-TYPE INTEGRAL EQUATION BY USING THE METHOD OF SUCCESSIVE APPROXIMATIONS

OBJECTIVES:

- To identify linear and non-linear Volterra integral equations
- To determine the solution of the Volterra linear integral equation
- To determine the solution of the Volterra non-linear integral equation

STRUCTURE:

3.1 Method of successive approximations for solving Volterra-type linear integral equations

3.2 Method of successive approximations for non-linear Volterra-type non-linear integral equation

3.3 Summary

3.4 Technical Terms

3.5 Self-Assessment Questions

3.6 Suggested Readings

3.1 METHOD OF SUCCESSIVE APPROXIMATIONS FOR SOLVING VOLTERRA TYPE LINER INTEGRAL EQUATION:

Suppose we have a Volterra-type integral equation of the second kind:

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt. \qquad (3.1)$$

We assume that $f(x)$ is continuous in $[0,a]$ and the kernel $K(x,t)$ is continuous for $0 \le x \le a,\ 0 \le t \le x$.

Take some function $\varphi_0(x)$ continuous in $[0,a]$.

Putting the function $\varphi_0(x)$ into the right side of (3.1) in place of $\varphi(x)$, we get

$$\varphi_1(x) = f(x) + \lambda \int_0^x K(x,t)\varphi_0(t)dt.$$

The thus defined function $\varphi_1(x)$ is also continuous in the interval $[0,a]$. Continuing the process, we obtain a sequence of functions

$$\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x), \cdots$$

where

$$\varphi_n(x) = f(x) + \lambda \int_0^x K(x,t)\varphi_{n-1}(t)dt.$$

Under the assumptions with respect to $f(x)$ and $K(x,t)$, the sequence $\{\varphi_n(x)\}$ converges, as $n \to \infty$, to the solution $\varphi(x)$ of the integral equation (3.1).

In particular, if for $\varphi_0(x)$ we take $f(x)$, then $\varphi_n(x)$ will be the partial sums of the series (2.2), of Lesson II, which defines the solution of the integral equation (3.1). A suitable choice of the "zero" approximation $\varphi_0(x)$ can lead to a rapid convergence of the sequence $\{\varphi_n(x)\}$ to the solution of the integral equation.

Example 3.1: Using the method of successive approximations, solve the integral

$$\varphi(x) = 1 + \int_0^x \varphi(t)\, dt$$

taking $\varphi_0(x) = 0$.

Solution: Since $\varphi_0(x) = 0$, it follows that

$$\varphi_1(x) = 1 + \int_0^x \varphi_0(t)\, dt = 1 + \int_0^x 0\, dt = 1.$$

Then

$$\varphi_2(x) = 1 + \int_0^x 1\, dt = 1 + x,$$

$$\varphi_3(x) = 1 + \int_0^x (1 + t)dt = 1 + x + \frac{x^2}{2},$$

$$\varphi_4(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right) dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Obviously

$$\varphi_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}.$$

Thus, $\varphi_n(x)$ is the n^{th} partial sum of the series

$$\Sigma_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Hence, it follows that

$$\varphi_n(x) \to e^x \text{ as } n \to \infty.$$

Thus, the function $\varphi(x) = e^x$ is a solution of the given integral equation.

Example 3.2: Using the method of successive approximations, solve the integral

$$\varphi(x) = x - \int_0^x (x - t)\, \varphi(t)\, dt\, , \varphi_0(x) = 0.$$

Solution:

Given the integral equation is

$$\varphi(x) = x - \int_0^x (x - t)\varphi(t)dt.$$

By using the method of successive approximation,

i.e. $\varphi_{n+1}(x) = f(x) + \lambda \int_0^x K(x,t)\varphi_n(t)\, dt$, $n = 0, 1, 2, \ldots$

Since $\varphi_0(x) = 0$. Then,

$$\varphi_1(x) = x - \int_0^x (x-t)\varphi_0(t)\, dt$$

$$= x$$

$$\varphi_2(x) = x - \int_0^x (x-t)\varphi_1(t)\, dt$$

$$= x - \int_0^x (x-t)\,(t)\, dt$$

$$= x - \frac{x^3}{3!}$$

$$\varphi_3(x) = x - \int_0^x (x-t)\,(t - \frac{t^3}{3!})\, dt$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Obviously,

$$\varphi_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Thus, $\varphi_n(x)$ is the n^{th} partial sum of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = sin\, x.$$

Hence, it follows that

$$\varphi_n(x) \to sin\, x \text{ as } n \to \infty.$$

Thus, the function $\varphi(x) = sin\, x$ is a solution of the given integral equation.

3.2 METHOD OF SUCCESSIVE APPROXIMATIONS FOR NON-LINEAR VOLTERRA-TYPE NON-LINER INTEGRAL EQUATION

The method of successive approximations can also be applied to the solution of nonlinear Volterra integral equations of the form

$$y(x) = y_0 + \int_0^x F[t, y(t)]dt \quad (3.2)$$

or the more general equations

$$\varphi(x) = f(x) + \int_0^x F\big(x, t, \varphi(t)\big)dt \quad (3.3)$$

under extremely broad assumptions with respect to the functions $F(x, t, z)$ and $f(x)$. The problem of solving the differential equation

$$\frac{dy}{dx} = F(x, y), \qquad y|_{x=0} = y_0$$

reduces to an equation of the type (3.2). As in the case of linear integral equations, we shall seek the solution of the equation (3.3) as the limit of the sequence $\{\varphi_n(x)\}$where, for example, $\varphi_0(x) = f(x)$, and the following elements $\varphi_k(x)$ are computed successively from the formula

$$\varphi_k(x) = f(x) + \int_0^x F(x, t, \varphi_{(k-1)}(t)dt, \ (k = 1, 2, \dots). \quad (3.4)$$

If $f(x)$ and $F(x, t, z)$ are quadratically summable and satisfy the conditions

$$|F(x, t, z_2) - F(x, t, z_1)| \le a(x, t)|z_2 - z_1| \quad (3.5)$$

$$\left|\int_0^x F(x, t, f(t))\, dt\right| \le n(x) \quad (3.6)$$

where the functions $a(x, t)$ and $n(x)$ are such that in the main domain $(0 \le t \le x \le a)$

$$\int_0^a n^2(x)dx \le N^2, \qquad \int_0^a dx \int_0^x a^2(x, t)dt \le A^2 \quad (3.7)$$

it follows that the nonlinear Volterra integral equation of the second kind (3.3) has a unique solution $\varphi(x) \in L_2(0, a)$ which is defined as the limit of $\varphi_n(x)$ as $n \to \infty$:

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$$

where the functions $\varphi_n(x)$ are found from the recursion formulas (3.4). For $\varphi_0(x)$ we can take any function in $L_2(0, a)$ (in particular, a continuous function), for which the condition (3.6) is fulfilled. Note that an apt choice of the zero approximation can facilitate solution of the integral equation.

Example 3.3: Using the method of successive approximations, solve the integral equation

$$\varphi(x) = \int_0^x \frac{1 + \varphi^2(t)}{1 + t^2}\, dt$$

taking as the zero approximation:

(a) $\varphi_0(x) = 0$, **(b)** $\varphi_0(x) = x$.

Solution. (a) Let $\varphi_0(x) = 0$. Then

$$\varphi_1(x) = \int_0^x \frac{1 + \varphi_0^2(t)}{1 + t^2}\, dt = \int_0^x \frac{1}{1 + t^2}\, dt = arctan\, x,$$

$$\varphi_2(x) = \int_0^x \frac{1 + \varphi_1^2(t)}{1 + t^2}\, dt = \int_0^x \frac{1 + arctan^2\, t}{1 + t^2}\, dt = arctan\, x + \frac{1}{3}\, arctan^3\, x$$

$$\varphi_3(x) = \int_0^x \frac{1 + \varphi_2^2(t)}{1 + t^2}\, dt = \int_0^x \frac{1 + \left(arctan\, t + \frac{1}{3} arctan^3 t\right)^2}{1 + t^2}\, dt$$

$$= arctan\, x + \frac{1}{3}\, arctan^3\, x + \frac{2}{3 \times 5}\, arctan^5\, x + \frac{1}{7 \times 9} arctan^7\, x$$

$$\varphi_4(x) = \int_0^x \frac{1 + \varphi_3^2(t)}{1 + t^2}\, dt$$

$$= \arctan x + \frac{1}{3}\, arctan^3\, x + \frac{2}{3 \times 5}\, arctan^5\, x + \frac{17}{5 \times 7 \times 9}\, arctan^7 x$$

$$+ \frac{38}{5 \times 7 \times 9^2}\, arctan^9 x + \frac{134}{9 \times 11 \times 21 \times 25}\, arctan^{11} x$$

$$+ \frac{4}{3 \times 5 \times 7 \times 9 \times 13}\, arctan^{13} x + \frac{1}{7^2 \times 9^2 \times 15}\, arctan^{15} x, \cdots.$$

Since $tanx = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \quad |x| < \frac{\pi}{2}$

we observe that

$$\varphi_n\,(x) \to\ tan(arctan\, x) = x \text{ as } n \to \infty.$$

Thus, the function $\varphi(x) = x$ is a solution of the given integral equation.

(b) Let $\varphi_0(x) = x$. Then

$$\varphi_1(x) = \int_0^x \frac{1 + \varphi_0^2(t)}{1 + t^2}\, dt = \int_0^x \frac{1 + t^2}{1 + t^2}\, dt = x$$

In a similar fashion, we find $\varphi_n(x) = x\ (n = 2, 3, \ldots)$.

Thus, the sequence $\{\varphi_n(x)\}$ is a stationary sequence $\{x\}$, the limit of which is $\varphi(x) = x$. The solution of this integral equation is obtained directly:

$\varphi(x) = x$.

3.3 SUMMARY:

In this lesson, find the solution of Volterra integral equations by using the method of successive approximations and the convolution theorem. Finally, we have given examples and self-assessment problems that we included for better understanding of the readers.

3.4 TECHNICAL TERMS:

Integral equation, linear, non-linear, Volterra integral equation, convolution theorem.

3.5 SELF-ASSESSMENT QUESTIONS:

(3a): Using the method of successive approximations, solve the following integral equations:

1. $\varphi(x) = x - \int_0^x (x-t)\varphi(t)\,dt$, $\varphi_0(x) = 0$.

2. $\varphi(x) = 1 - \int_0^x (x-t)\varphi(t)dt$, $\varphi_0(x) = 0$.

3. $\varphi(x) = 1 + \int_0^x (x-t)\varphi(t)\,dt$, $\varphi_0(x) = 1$.

4. $\varphi(x) = x + 1 - \int_0^x \varphi(t)\,dt$;

   $(a)\ \varphi_0(x) = 1,\ (b)\ \varphi_0(x) = x+1.$

5. $\varphi(x) = \frac{x^2}{2} + x - \int_0^1 \varphi(t)\,dt$;

   $(a)\ \varphi_0(x) = 1, \quad (b)\ \varphi_0(x) = x, \quad (c)\ \varphi_0(x) = \frac{x^2}{2} + x.$

6. $\varphi(x) = 1 + x + \int_0^x (x-t)\varphi(t)dt$, $\varphi_0(x) = 1$.

7. $\varphi(x) = 2x + 2 - \int_0^x \varphi(t)\,dt$;

   $(a)\ \varphi_0(x) = 1,\ (b)\ \varphi_0(x) = 2.$

9. $\varphi(x) = 2x^2 + 2 - \int_0^x x\,\varphi(t)\,dt$;

   $(a)\ \varphi_0(x) = 2,\ (b)\ \varphi_0(x) = 2x.$

10. $\varphi(x) = \frac{x^3}{3} - 2x - \int_0^x \varphi(t)dt$, $\varphi_0(x) = x^2$.

(3b):

1. Using the method of successive approximations to solve the following integral equations

$$\varphi(x) = \int_0^x \frac{t\varphi(t)}{1+t+\varphi(t)}\,dt.$$

2. Using the method of successive approximations to find a second approximation $\varphi_2(x)$ to the solution of the integral equation

$$\varphi(x) = 1 + \int_0^x [\varphi^2(t) + t\varphi(t) + t^2]\,dt.$$

3. Using the method of successive approximations to find a third approximation $\varphi_3(x)$ to the solution of the integral equation

$$\varphi(x) = \int_0^x [t\varphi^2(t) - 1]\ dt.$$

Solutions to Self-Assessment Questions:

Exercise (3a):

1. $\varphi(x) = sin\, x$
2. $\varphi(x) = cos\, x$
3. $\varphi(x) = cosh\, x$
4. $\varphi(x) = 1$
5. $\varphi(x) = x$
6. $\varphi(x) = e^x$
7. $\varphi(x) = 2$
8. $\varphi(x) = 2$
9. $\varphi(x) = x^2 - 2x$

Exercise (3b):

1. $\varphi(x) \equiv 0$
2. $\varphi_2\,(x) = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{13}{24}x^4 + \frac{1}{4}x^5 + \frac{1}{18}x^6 + \frac{1}{63}x^7$
3. $\varphi_3(x) = -x + \frac{x^4}{4} - \frac{x^7}{14} + \frac{x^{10}}{160}$

3.6 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971, by M. Krasnov, A. Kiselev and G. Makarenko.
2. Integral equations, Krishna's Educational Publishers, Meerut- 250001, India, 1975, by Shanti Swarup and Shiv Raj Singh.
3. Integral Equations and Boundary Value Problems, S. Chand & Company PVT. LTD, New Delhi-110055, India, 2007, by Dr. M.D. Raisinghania.
4. Integral Equations and their Applications, WIT press, 25 Bridge Street, Billerica, MA 01821, USA, by M. Rahman.
5. Introduction to Integral Equations with Applications, John Wiley & Sons, 1999, by Jerri, A.
6. Linear Integral Equation, Theory and Techniques, Academic Press, 2014, by Kanwal R. P.

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LESSON- 4

SOLUTION OF VOLTERRA-TYPE INTEGRAL EQUATION BY USING CONVOLUTION THEOREM

OBJECTIVES:

- To apply the convolution theorem in Laplace transformations to an integral equation to determine the solution of the Volterra integral equation.
- To apply the convolution theorem in Laplace transformations to integral equations to determine the solution for a system of Volterra integral equations.

STRUCTURE:

4.1 Solution of Volterra Integral Equation by using the Convolution Theorem.

4.2 Solution of System of Volterra Integral Equations by using the Convolution Theorem.

4.3 Summary

4.4 Technical Terms

4.5 Self-Assessment Questions

4.6 Suggested Readings

4.1 SOLUTION OF VOLTERRA INTEGRAL EQUATION BY USING THE CONVOLUTION THEOREM:

Let $\varphi_1(x)$ and $\varphi_2(x)$ be two continuous functions defined for $x \geq 0$. The convolution of these two functions is the function $\varphi_3(x)$ defined by the equation

$$\varphi_3(x) = \int_0^x \varphi_1(x-t)\varphi_2(t)\,dt. \quad (4.1)$$

This function, defined for $x \geq 0$, will also be a continuous function. If $\varphi_1(x)$ and $\varphi_2(x)$ are original functions for the Laplace transformation, then

$$L[\varphi_3] = L[\varphi_1] \cdot L[\varphi_2] \quad (4.2)$$

i.e., the transform of a convolution is equal to the product of the transforms of the functions (convolution theorem). Let us consider the Volterra-type integral equation of the second kind

$$\varphi(x) = f(x) + \int_0^x K(x-t)\varphi(t)\,dt \quad (4.3)$$

the kernel of which is dependent solely on the difference $x - t$. We shall call equation (4.3) an integral equation of the convolution type.

Let $f(x)$ and $K(x)$ be sufficiently smooth functions which, as $x \to \infty$, do not grow faster than the exponential function, so that

$$|f(x)| \leq M_1 e^{s_1 x}, \qquad |K(x)| \leq M_2 e^{s_2 x}. \qquad (4.4)$$

Applying the method of successive approximations, we can show that in this case, the function $\varphi(x)$ will also satisfy an upper bound of type (4.4):

$$|\varphi(x)| \leq M_3 e^{s_3 x}.$$

Consequently, the Laplace transform of the functions $f(x), K(x)$ and $\varphi(x)$ can be found (it will be defined in the half-plane $Re\ p = s > max\ (s_1, s_2, s_3)$).

Let

$$L[f(x)] = F(p), \qquad L[\varphi(x)] = \phi(p), \qquad L[K(x)] = \widetilde{K}(p).$$

Taking the Laplace transform of both sides of (4.3) and employing the convolution theorem, we find

$$\phi(p) = F(p) + \widetilde{K}(p)\phi(p). \qquad (4.5)$$

Hence,

$$\phi(p) = \frac{F(p)}{1 - \widetilde{K}(p)}, (\widetilde{K}(p) \neq 1).$$

Apply the inverse Laplace transform to both sides, we get the solution of the integral equation (4.3).

Example 4.1:

Solve the integral equation

$$\varphi(x) = sin\, x + 2\int_0^x cos(x-t)\varphi(t)\ dt.$$

Solution. Given that the integral equation is

$$\varphi(x) = sin\, x + 2\int_0^x cos(x-t)\varphi(t)\ dt \qquad (4.6)$$

where $f(x) = sinx, \lambda = 2, K(x,t) = K(x-t) = \cos(x-t)$.

Apply the Laplace transform on both sides of the equation (4.6) and taking account of the convolution theorem (transform of a convolution), we get

$\mathrm{L}[\varphi(x)] = L[sin\, x] + 2\ (L[\cos(x)] \cdot L[\varphi(x)])$

$$\phi(p) = \frac{1}{p^2+1} + \frac{2p}{p^2+1}\phi(p)$$

$$\phi(p)\left[1 - \frac{2p}{p^2+1}\right] = \frac{1}{p^2+1}$$

(or)

$$\phi(p) = \frac{1}{(p-1)^2}.$$

Apply the Laplace inverse transformation on both sides

$$L^{-1}[\phi(p)] = L^{-1}\left[\frac{1}{(p-1)^2}\right]$$

$$\varphi(x) = e^x L^{-1}\left[\frac{1}{p^2}\right] \text{ (by using the shifting operator)}$$

$$\varphi(x) = xe^x.$$

Hence, the solution of the given integral equation is

$$\varphi(x) = xe^x.$$

Example 4.2:

Solve the integral equation

$$\varphi(x) = e^x - \int_0^x e^{x-t}\,\varphi(t)\,dt.$$

Solution. Given that the integral equation is

$$\varphi(x) = e^x - \int_0^x e^{x-t}\,\varphi(t)\,dt \qquad (4.7)$$

where $f(x) = e^x, \lambda = -1, K(x,t) = K(x-t) = e^{x-t}$.

Apply the Laplace transform on both sides of the equation (4.7) and taking account of the convolution theorem (transform of a convolution), we get

$$L[\varphi(x)] = L[e^x] - L\,[e^x]\cdot L[\varphi(x)]$$

$$\phi(p) = \frac{1}{p-1} - \frac{1}{p-1}\phi(p)$$

$$\phi(p)\left[1 + \frac{1}{p-1}\right] = \frac{1}{p-1}$$

(or)

$$\phi(p) = \frac{1}{p}.$$

Apply the Laplace inverse transform on both sides

$$L^{-1}[\phi(p)] = L^{-1}\left[\frac{1}{p}\right]$$

$$\varphi(x) = 1.$$

Hence, the solution of the given integral equation is

$$\varphi(x) = 1.$$

4.2: SOLUTION OF THE SYSTEM OF VOLTERRA INTEGRAL EQUATIONS BY USING THE CONVOLUTION THEOREM:

The Laplace transformation method may be employed in finding the solution of a system of Volterra integral equations of the type

$$\varphi_i(x) = f_i(x) + \sum_{j=1}^{s} \int_0^x K_{ij}(x-t)\,\varphi_j(t)dt \quad (i = 1, 2, \cdots, s) \qquad (4.8)$$

where $K_{ij}(x)$, $f_i(x)$ are known continuous functions having Laplace transforms. Taking the Laplace transform of both sides of (4.8), we get

$$\phi_i(p) = F_i(p) + \sum_{j=1}^{s} \widetilde{K}_{ij}(p)\phi_j(p) \quad (i = 1, 2, \cdots, s) \qquad (4.9)$$

This is a system of linear algebraic equations in $\phi_j(p)$. Solving it, we find $\phi_j(p)$, the original functions of which will be the solution of the original system of integral equations (4.8).

Example 4.3: Solve the system of integral equations,

$$\varphi_1(x) = 1 - 2\int_0^x e^{2(x-t)}\varphi_1(t)dt + \int_0^x \varphi_2(t)dt,$$

$$\varphi_2(x) = 4x - \int_0^x \varphi_1(t)dt + 4\int_0^x (x-t)\,\varphi_2(t)dt.$$

Solution. Given the system of integral equations

$$\varphi_1(x) = 1 - 2\int_0^x e^{2(x-t)}\varphi_1(t)dt + \int_0^x \varphi_2(t)dt, \qquad (4.10)$$

$$\varphi_2(x) = 4x - \int_0^x \varphi_1(t)dt + 4\int_0^x (x-t)\,\varphi_2(t)dt. \qquad (4.11)$$

Equations (4.10) and (4.11) can be written as

$$\varphi_1(x) = 1 - 2\int_0^x e^{2(x-t)}\varphi_1(t)dt + \int_0^x (x-t)^0\varphi_2(t)dt, \qquad (4.12)$$

$$\varphi_2(x) = 4x - \int_0^x (x-t)^0\varphi_1(t)dt + 4\int_0^x (x-t)\,\varphi_2(t)dt. \qquad (4.13)$$

Applying the Laplace transform on both sides and using the convolution theorem for equations (4.12) and (4.13) respectively, we get:

$$\phi_1(p) = \frac{1}{p} - \frac{2}{p-2}\,\phi_1(p) + \frac{1}{p}\,\phi_2(p)$$

$$\phi_1(p)\left[1 + \frac{2}{p-2}\right] - \frac{1}{p}\,\phi_2(p) = \frac{1}{p}$$

$$\Rightarrow p^2\,\phi_1(p) - (p-2)\phi_2(p) = p-2, \qquad (4.14)$$

$$\phi_2(p) = \frac{4}{p^2} - \frac{1}{p}\,\phi_1(p) + \frac{4}{p^2}\,\phi_2(p)$$

$$\frac{1}{p}\,\phi_1(p) + \left[1 - \frac{4}{p^2}\right]\phi_2(p) = \frac{4}{p^2}$$

$$\Rightarrow p\,\phi_1(p) + (p^2 - 4)\phi_2(p) = 4. \qquad (4.15)$$

Solving the equations (4.14) and (4.15), we get:

$$\phi_1(p) = \frac{p}{(p+1)^2} = \frac{1}{p+1} - \frac{1}{(p+1)^2}, \qquad (4.16)$$

$$\phi_2(p) = \frac{3p+2}{(p-2)(p+1)^2} = \frac{8}{9}\cdot\frac{1}{p-2} + \frac{1}{3}\cdot\frac{1}{(p+1)^2} - \frac{8}{9}\cdot\frac{1}{p+1}. \qquad (4.17)$$

Apply the Laplace inverse transform on both sides of the equation (4.16), we get

$$L^{-1}[\phi_1(p)] = L^{-1}\left[\frac{1}{p+1}\right] - L^{-1}\left[\frac{1}{(p+1)^2}\right]$$

$$\varphi_1(x) = e^{-x} - e^{-x}L^{-1}\left[\frac{1}{p^2}\right]$$

$$\varphi_1(x) = e^{-x} - xe^{-x}.$$

Apply the Laplace inverse transform on both sides of the equation (4.17), we get

$$L^{-1}[\phi_2(p)] = \frac{8}{9}\cdot L^{-1}\left[\frac{1}{p-2}\right] + \frac{1}{3}\cdot L^{-1}\left[\frac{1}{(p+1)^2}\right] - \frac{8}{9}\cdot L^{-1}\left[\frac{1}{p+1}\right]$$

$$\varphi_2(x) = \frac{8}{9}\cdot e^{2x} + \frac{1}{3}\cdot e^{-x}L^{-1}\left[\frac{1}{p^2}\right] - \frac{8}{9}\cdot e^{-x}$$

$$\varphi_2(x) = \frac{8}{9}\cdot e^{2x} + \frac{1}{3}x\cdot e^{-x} - \frac{8}{9}\cdot e^{-x}.$$

The functions $\varphi_1(x)$, $\varphi_2(x)$ are solutions of the system of integral equations (4.10) and (4.11) respectively.

4.3 SUMMARY:

In this lesson, we find the solution to the Volterra integral equation by using the convolution theorem. Next, we find the solution to the system of Volterra integral equations by using the convolution theorem. Finally, we have given examples and self-assessment problems that we included for better understanding of the readers.

4.4 TECHNICAL TERMS:

Integral equation, Volterra integral equation, Laplace transformation, inverse Laplace transformation, convolution theorem, system of equations.

4.5 SELF-ASSESSMENT QUESTIONS:

(4a): Solve the following integral equations:

1. $\varphi(x) = x - \int_0^x e^{x-t}\varphi(t)dt.$
2. $\varphi(x) = e^{2x} + \int_0^x e^{t-x}\varphi(t)dt.$
3. $\varphi(x) = x - \int_0^x (x-t)\varphi(t)dt.$
4. $\varphi(x) = cos(x) - \int_0^x (x-t)cos(x-t)\varphi(t)dt.$
5. $\varphi(x) = 1 + x + \int_0^x e^{-2(x-t)}\varphi(t)dt.$
6. $\varphi(x) = x + \int_0^x sin(x-t)\varphi(t)dt.$
7. $\varphi(x) = sin(x) + \int_0^x (x-t)\varphi(t)dt.$
8. $\varphi(x) = x - \int_0^x \sinh(x-t)\,\varphi(t)dt.$
9. $\varphi(x) = 1 - 2x - 4x^2 + \int_0^x [3 - 6(x-1) - 4(x-1)^2]\,\varphi(t)\,dt.$
10. $\varphi(x) = sinh\,x - \int_0^x cosh(x-1)\,\varphi(t)\,dt.$
11. $\varphi(x) = 1 + 2\int_0^x cos(x-1)\,\varphi(t)\,dt.$
12. $\varphi(x) = e^x + 2\int_0^x cos(x-1)\,\varphi(t)\,dt.$
13. $\varphi(x) = cosx + \int_0^x \varphi(t)\,dt.$

(4b): Solve the following systems of integral equations:

1. $\varphi_1(x) = sinx + \int_0^x \varphi_2(t)dt,$

   $\varphi_2(x) = 1 - cosx - \int_0^x \varphi_1(t)dt.$

2. $\varphi_1(x) = e^{2x} + \int_0^x \varphi_2(t)dt,$

   $\varphi_2(x) = 1 - \int_0^x e^{2(x-t)}\varphi_1(t)dt.$

3. $\varphi_1(x) = e^x + \int_0^x \varphi_1(t)dt - \int_0^x e^{x-t}\varphi_2(t)dt,$

   $\varphi_2(x) = -x - \int_0^x (x-t)\varphi_1(t)dt + \int_0^x \varphi_2(t)dt.$

4. $\varphi_1(x) = e^x - \int_0^x \varphi_1(t)dt + 4\int_0^x e^{x-t}\varphi_2(t)dt,$

   $\varphi_2(x) = 1 - \int_0^x e^{t-x}\varphi_1(t)dt + \int_0^x \varphi_2(t)dt.$

5. $\varphi_1(x) = x + \int_0^x \varphi_2(t)dt,$

   $\varphi_2(x) = 1 - \int_0^x \varphi_1(t)dt,$

   $$\varphi_3(x) = sinx + \frac{1}{2}\int_0^x (x-t)\,\varphi_1(t)dt.$$

6. $\varphi_1(x) = 1 - \int_0^x \varphi_2(t)dt,$

$$\varphi_2(x) = cosx - 1 + \int_0^x \varphi_3(t)dt,$$

$$\varphi_3(x) = cosx + \int_0^x \varphi_1(t)dt,$$

7. $\varphi_2(x) = x + 1 + \int_0^x \varphi_3(t)dt,$

$$\varphi_2(x) = -x + \int_0^x (x-t)\,\varphi_1(t)dt.$$

$$\varphi_2(x) = cosx - 1 - \int_0^x \varphi_1(t)dt.$$

Solutions to Self-Assessment Questions:

Exercise (4a):

1. $\varphi(x) = x - \frac{x^2}{2}$
2. $\varphi(x) = \frac{1}{2}(3e^{2x} - 1)$
3. $\varphi(x) = sin\, x$
4. $\varphi(x) = \frac{1}{3}\left(2\, cos\, \sqrt{3x} + 1\right)$
5. $\varphi(x) = 2x + 1$
6. $\varphi(x) = x + \frac{x^3}{6}$
7. $\varphi(x) = \frac{1}{2} sin\, x + \frac{1}{2} sinh\, x$
8. $\varphi(x) = x - \frac{x^3}{6}$
9. $\varphi(x) = e^x$
10. $\varphi(x) = \frac{2}{\sqrt{5}} \sinh \frac{\sqrt{5}}{2}\ x \cdot e^{-\frac{x}{2}}$
11. $\varphi(x) = 1 + 2xe^x$
12. $\varphi(x) = e^x(1 + x)^2$
13. $\varphi(x) = \varphi(x) = \frac{e^x + cos\, x + sin\, x}{2}$

Exercise (4b):

1. $\varphi_1(x) = sin\, x, \varphi_2(x) = 0$
2. $\varphi_1(x) = 3e^x - 2, \varphi_2(x) = 3e^x - 2e^{2x}$

3. $\varphi_1(x) = e^{2x}, \varphi_2(x) = \frac{1-e^{2x}}{2}$
4. $\begin{cases} \varphi_1\,(x) = (x+2)sin\,x + (2x+1)cos\,x, \\ \varphi_2\,(x) = \frac{2+x}{2}\,cos\,x - \frac{2x+1}{2}\,sin\,x \end{cases}$
5. $\varphi_1(x) = 2\,sin\,x,\ \varphi_2(x) = 2\,cos\,x - 1, \varphi_3(x) = x$
6. $\varphi_1(x) = cos\,x, \varphi_2(x) = sin\,x, \varphi_3(x) = s\,in\,x + cos\,x$
7. $\begin{cases} \varphi_1(x) = \left(1+\frac{x}{2}\right)cos\,x + \frac{1}{2}\,cosx, \\ \varphi_2(x) = 1 - x + \frac{1}{2}sinx - \left(1+\frac{x}{2}\right)cosx, \\ \varphi_3(x) = cos\,x - 1 - \left(1+\frac{x}{2}\right)sinx \end{cases}$

4.6 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971, by M. Krasnov, A. Kiselev and G. Makarenko.
2. Integral equations, Krishna’s Educational Publishers, Meerut- 250001, India, 1975, by Shanti Swarup and Shiv Raj Singh.
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LESSON- 5

INTEGRO-DIFFERENTIAL EQUATIONS

OBJECTIVES:

- To learn about the integro-differential equations
- To determine the solutions to the integro-differential equations using Laplace transforms

STRUCTURE:

5.1 Introduction

5.2 Integro-differential equations

5.3 Solution of Integro-differential equations with the aid of the Laplace Transformations

5.4 Summary

5.5 Technical Terms

5.6 Self-Assessment Questions

5.7 Suggested Readings

5.1. INTRODUCTION:

This lesson deals with one of the most applied problems in the engineering sciences. It concerns integro-differential equations, where both differential and integral operators appear in the same equation. Volterra introduced this type of equation for the first time in the early 1900s. Volterra investigated population growth, focusing his study on hereditary influences, where, through his research work, the topic of integro-differential equations was established.

5.2. INTEGRO-DIFFERENTIAL EQUATIONS:

5.2.1. Definition: An integral equation is an equation in which an unknown function to be determined appears under one or more integral signs. If the derivatives of the function are involved, it is called an integro-differential equation.

$$\varphi^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt,$$

where $\varphi^{(n)}(x) = \frac{d^n\varphi}{dx^n}$, and $K(x,t)$ be the kernel.

The above equation is the combination of a differential operator and an integral operator; therefore, it is necessary to define initial conditions $\varphi(0), \varphi'(0), \dots, \varphi^{(n-1)}(0)$ for the determination of the particular solution $\varphi(x)$ of the integro-differential equation.

5.2.2. Note: There are mainly two types of linear integro-differential equations:

(i) Volterra integro-differential equation

(ii) Fredholm integro-differential equation

5.2.3. Definition: A Volterra integro-differential equation is defined as if the upper limit of the integral of the integro-differential equation is a variable.

5.2.4. Definition: Fredholm integro-differential equation is defined as if the limits of the integral of the integro-differential equation are fixed constants.

5.2.5. Examples: Consider the following examples,

$$\varphi'(x) = f(x) - \int_0^x (x-t)\varphi(t)dt, \quad \varphi(0) = 0 \qquad (1)$$

$$\varphi''(x) = g(x) + \int_0^x (x-t)\varphi(t)dt, \quad \varphi(0) = 0, \varphi'(0) = -1 \qquad (2)$$

$$\varphi'(x) = e^x - x + \int_0^1 xt\varphi(t)dt, \quad \varphi(0) = 0 \qquad (3)$$

$$\varphi''(x) = h(x) + \int_0^1 t\varphi'(t)dt, \quad \varphi(0) = 0, \varphi'(0) = 1 \qquad (4)$$

It is clear from the above examples that the unknown function $\varphi(x)$ or one of its derivatives appears under the integral sign, and the other derivatives appear outside the integral sign as well. Equations (1) and (2) are Volterra-type integro-differential equations, and equations (3) and (4) are Fredholm-type integro-differential equations. It is to be noted that these equations are linear integro-differential equations. However, nonlinear integro-differential equations also arise in many scientific and engineering problems. To obtain a solution of the integro-differential equation, we need to specify the initial conditions to determine the unknown constants.

5.2.6. Note: One quick source of integro-differential equations can be clearly seen when we convert the differential equation to an integral equation by using Leibniz rule, i.e.,

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,t)dt = \int_{a(x)}^{b(x)} \frac{\partial F(x,t)}{\partial x} dt + \frac{db(x)}{dx} F\big(x, b(x)\big) - \frac{da(x)}{dx} F(x, a(x))$$

5.2.7. Note: In the electrical engineering problem, the current $I(t)$ flowing in a closed circuit can be obtained in the form of the following integro-differential equation,

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_0^t I(\tau)d\tau = f(t), \qquad I(0) = I_0$$

where, L is the inductance, R the resistance, C the capacitance, and $f(t)$ the applied voltage.

5.2.8. Definition: If the kernel $K(x,t)$ of the integral equation is defined as a function of the difference (x,t), i.e.,

$$K(x,t) = K(x-t)$$

where K is a certain function of one variable.

Now, we recall some important definitions and properties of the Laplace and inverse Laplace transforms, which are essential for the study of further concepts.

5.2.9. Laplace Transformation and its properties:

Definition: Consider a function φ in terms of x and its Laplace transformation will be a function Φ in terms of p i.e., $L\{\varphi(x)\} = \Phi(p)$ (or) $\varphi(x) \risingdotseq \Phi(p)$.

Properties: If $L\{\varphi(x)\} = \Phi(p)$ then,

(1) $L\{e^{ax}\varphi(x)\} = \Phi(p-a)$,

(2) $L\{e^{-ax}\varphi(x)\} = \Phi(p+a)$

(3) $L\{\varphi(ax)\} = \frac{1}{a}\Phi(\frac{p}{a})$

(4) $L\{\varphi'(x)\} = pL\{\varphi(x)\} - \varphi(0)$

(5) $L\{\varphi''(x)\} = p^2L\{\varphi(x)\} - p\varphi(0) - \varphi'(0)$

(6) $L\{x\varphi(x)\} = \frac{-d}{dp}\Phi(p)$

(7) $L\{x^n\varphi(x)\} = (-1)^n \frac{d^n}{dp^n}\Phi(p)$

(8) The convolution of two functions $\varphi(x)$ and $\psi(x)$ is defined as, $\varphi(x) * \psi(x) = \int_0^x \varphi(t)\,\psi(x-t)dt = \int_0^x \varphi(x-t)\,\psi(t)dt$

(9) $L\{1\} = \frac{1}{p}$

(10) $L\{sinax\} = \frac{a}{a^2+p^2}$

(11) $L\{cosax\} = \frac{p}{a^2+p^2}$

(12) $L\{x\} = \frac{1}{p^2}$

(13) $L\{e^{ax}\} = \frac{1}{p-a}$

(14) $L\{sinhax\} = \frac{a}{p^2-a^2}$

(15) $L\{coshax\} = \frac{p}{p^2-a^2}$

(16) $L\left\{\frac{xsinax}{2a}\right\} = \frac{p}{(p^2+a^2)^2}$

(17) $L\{x\,e^{ax}\} = \frac{1}{(p-a)^2}$

5.2.10. Inverse Laplace Transformation and its properties:

Consider a function Φ in terms of p then its inverse Laplace transformation will be φ in terms of x, i.e., $L^{-1}\{\Phi(p)\} = \varphi(x)$

Properties: If $L^{-1}\{\Phi(p)\} = \varphi(x)$, then,

(18) $L^{-1}\left\{\frac{1}{p}\right\} = 1$

(19) $L^{-1}\left\{\frac{a}{a^2+p^2}\right\} = sinax$

(20) $L^{-1}\left\{\frac{p}{a^2+p^2}\right\} = cosax$

(21) $L^{-1}\left\{\frac{1}{p^2}\right\} = x$

(22) $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{ax}$

(23) $L^{-1}\left\{\frac{a}{p^2-a^2}\right\} = sinhax$

(24) $L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = coshax$

(25) $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \frac{xsinax}{2a}$

(26) $L^{-1}\left\{\frac{1}{(p-a)^2}\right\} = x\, e^{ax}$

(27) If $L\{\varphi(x)\} = \Phi(p)$ and $L\{\psi(x)\} = \Psi(p)$ then,

$$L^{-1}\{\varphi(x)\psi(x)\} = \int_0^x \varphi(x-t)\, \psi(t)dt \; = \; \varphi * \psi$$

(or)

$$L^{-1}\{\varphi(x)\psi(x)\} = \int_0^x \varphi(t)\, \psi(x-t)dt \; = \; \varphi * \psi$$

known as the convolution theorem.

(28) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{x^n}{n!}$

5.3 SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS WITH THE AID OF THE LAPLACE TRANSFORMATIONS:

A linear integro-differential equation is an equation of the form

$$a_0(x)\varphi^n(x) + a_1(x)\varphi^{n-1}(x) + \cdots + a_n(x)\varphi(x) + \sum_{m=0}^{s} \int_0^x K_m(x,t)\varphi^{(m)}(t)dt = f(x) \quad (1)$$

Here,$a_0(x)$,...,$a_n(x), f(x), K_m(x,t)$ $(m = 0,1,2,\dots,s)$are known functions and $\varphi(x)$ is the unknown function. Unlike the case of integral equations, when solving integro-differential equations (1), initial conditions of the form

$$\varphi(0) = \varphi_{0,}\ \varphi'(0) = \varphi_0',\ \dots, \varphi^{(n-1)}(0) = \varphi_0^{(n-1)} \quad (2)$$

are imposed on the unknown function $\varphi(x)$. In (1), let the coefficients $a_k(x) =$ constant $(k = 0,1,\dots,n)$ and let $K_m(x,t) = K_m(x-t)(m = 0,1,\dots,s)$, that is, all the K_m depend solely on the difference $(x-t)$ of arguments. Without loss of generality, we can take $a_0 = 1$. Then equation (1) assumes the form

$$\varphi^n(x) + a_1\varphi^{(n-1)}(x) + \cdots + a_n\varphi(x) + \sum_{m=0}^{s} \int_0^x K_m(x-t)\varphi^{(m)}(t)dt = f(x) \quad (3)$$

where $a_1,\dots,a_n$ are constants.

Also, let the functions $f(x)$ and $K_m(x)$ be original functions,

$$f(x) \risingdotseq F(p),\ K_m(x) \risingdotseq \widetilde{K}_m(p) \quad (m = 0,1,2,\dots,s)$$

Then the function $\varphi(x)$ will also have the Laplace transform

$$\varphi(x) \risingdotseq \Phi(p)$$

Take the Laplace transform of both sides of (3). By virtue of the theorem on the transform of a derivative,

$$\varphi^{(k)}(x) \risingdotseq p^k\Phi(p) - p^{k-1}\varphi_0 - p^{k-2}\varphi_0' - \cdots - \varphi_0^{(k-1)} \quad (k = 0,1,2,\dots,n) \qquad (4)$$

By the convolution theorem,

$$\int_0^x K_m(x-t)\varphi^{(m)}(t)dt \risingdotseq \widetilde{K}_m(p)\left[p^m\Phi(p) - p^{m-1}\varphi_0 - \cdots - \varphi_0^{(m-1)}\right] \quad (m = 0,1,2,\dots,s) \qquad (5)$$

Equation (3) will therefore become

$$\Phi(p)\left[p^n + a_1p^{n-1} + \cdots + a_n + \sum_{m=0}^{s}\widetilde{K}(p)\,p^m\right] = A(p) \qquad (6)$$

where, $A(p)$ is some known function of p.

From (6) we find $\Phi(p)$, which is an operator solution of the problem. Finding the original function for $\Phi(p)$, we get the solution $\varphi(x)$ of the integro-differential equation (3) that satisfies the initial conditions (2).

5.3.1. Example: Solve the integro-differential equation,

$$\varphi''(x) + \int_0^x e^{2(x-t)}\varphi'(t)dt = e^{2x}, \quad \varphi(0) = \varphi'(0) = 0$$

by using the Laplace Transformation.

Solution: Consider the given integro-differential equation,

$$\varphi''(x) + \int_0^x e^{2(x-t)}\varphi'(t)dt = e^{2x}, \quad \varphi(0) = \varphi'(0) = 0$$

Let us take the Laplace Transformation defined as, $L\{\varphi(x)\} = \Phi(p)$

Also, we have,

$$L\{\varphi'(x)\} = pL\{\varphi(x)\} - \varphi(0)$$

$$L\{\varphi''(x)\} = p^2L\{\varphi(x)\} - p\varphi(0) - \varphi'(0)$$

From the given conditions, $\varphi(0) = \varphi'(0) = 0$,

the above two equations will become,

$$L\{\varphi'(x)\} = pL\{\varphi(x)\}$$
$$= p\Phi(p)$$
$$L\{\varphi''(x)\} = p^2L\{\varphi(x)\} - p\varphi(0) - \varphi'(0)$$
$$= p^2\Phi(p)$$

Consider the given equation and apply the Laplace Transformation on both sides,

$$L\{\varphi''(x)\} + L\left\{\int_0^x e^{2(x-t)}\varphi'(t)dt\right\} = L\{e^{2x}\}$$

$$L\{\varphi''(x)\} + L\{e^{2x} * \varphi'(x)\} = L\{e^{2x}\} \qquad \text{[Property-8]}$$

$$L\{\varphi''(x)\} + L\{e^{2x}\} * L\{\varphi'(x)\} = L\{e^{2x}\}$$

Now, substitute the above values in this equation, we get,

$$p^2\Phi(p) + \left(\frac{1}{p-2}\right)p\Phi(p) = \frac{1}{p-2}$$

$$\Phi(p)\left[\frac{p(p-1)^2}{p-2}\right]=\frac{1}{p-2}$$

$$\Phi(p)=\frac{1}{p(p-1)^2}$$

$$L\{\varphi(x)\}=\frac{1}{p(p-1)^2}$$

$$\varphi(x)=L^{-1}\left\{\frac{1}{p(p-1)^2}\right\}$$

By using partial fractions, we get,

$$\varphi(x)=L^{-1}\left\{\frac{1}{p}+\frac{1}{p-1}+\frac{1}{(p-1)^2}\right\}$$

$$\varphi(x)=L^{-1}\left\{\frac{1}{p}\right\}+L^{-1}\left\{\frac{1}{p-1}\right\}+L^{-1}\left\{\frac{1}{(p-1)^2}\right\}$$

$$\varphi(x)=1-e^x+xe^x \qquad \text{[Properties-18, 22, 26]}$$

Hence, this is the required solution for the given integro-differential equation.

5.3.2. Example: Solve the integro-differential equation,

$$\varphi'(x)-\varphi(x)+\int_0^x (x-t)\varphi'(t)dt-\int_0^x \varphi(t)dt=x; \quad \varphi(0)=-1,$$

by using the Laplace Transformation.

Solution: Consider the given integro-differential equation,

$$\varphi'(x)-\varphi(x)+\int_0^x (x-t)\varphi'(t)dt-\int_0^x \varphi(t)dt=x; \quad \varphi(0)=-1$$

Let us take the Laplace Transformation defined as, $L\{\varphi(x)\}=\Phi(p)$

Also, we have,

$$L\{\varphi'(x)\}=pL\{\varphi(x)\}-\varphi(0)$$

From the given condition, $\varphi(0)=-1$, the above equation will become,

$$L\{\varphi'(x)\}=pL\{\varphi(x)\}+1$$

$$=p\Phi(p)+1$$

Consider the Laplace Transformation on both sides,

$$L\{\varphi'(x)\}-\mathrm{L}\{\varphi(x)\}+L\{\int_0^x (x-t)\varphi'(t)dt\}-L\{\int_0^x \varphi(t)dt\}=L\{x\}$$

$$L\{\varphi'(x)\}-\mathrm{L}\{\varphi(x)\}+L\{x*\varphi'(x)\}-L\{1*\varphi(x)\}=L\{x\}$$

[Property-8]

$$L\{\varphi'(x)\}-\mathrm{L}\{\varphi(x)\}+L\{x\}L\{\varphi'(x)\}-L\{1\}L\{\varphi(x)\}=L\{x\}$$

Now, substitute the above values in this equation, we get,

$$p\Phi(p)+1-\Phi(p)+\frac{1}{p^2}[p\Phi(p)+1]-\frac{1}{p}\Phi(p)=\frac{1}{p^2}$$

$$p\Phi(p)+1-\Phi(p)+\frac{\Phi(p)}{p}+\frac{1}{p^2}-\frac{\Phi(p)}{p}=\frac{1}{p^2}$$

$$p\Phi(p)+1-\Phi(p)=0$$

$$\Phi(p)[p-1] = -1$$
$$\Phi(p) = \frac{-1}{p-1}$$
$$L\{\varphi(x)\} = \frac{-1}{p-1}$$
$$\varphi(x) = L^{-1}\left\{\frac{-1}{p-1}\right\}$$
$$\varphi(x) = -e^{x}$$ [Property-22]

Hence, this is the required solution for the given integro-differential equation.

5.3.3. Example: Solve the integro-differential equation,

$\varphi''(x) - 2\varphi'(x) + \varphi(x) + 2\int_0^x \cos(x-t)\,\varphi''(t)dt + 2\int_0^x \sin(x-t)\,\varphi'(t)dt = cosx;$

$\varphi(0) = \varphi'(0) = 0$, by using the Laplace Transformation.

Solution: Consider the given integro-differential equation,

$\varphi''(x) - 2\varphi'(x) + \varphi(x) + 2\int_0^x \cos(x-t)\,\varphi''(t)dt + 2\int_0^x \sin(x-t)\,\varphi'(t)dt =$

$cosx;\;\; \varphi(0) = \varphi'(0) = 0$

Let us take the Laplace Transformation defined as, $L\{\varphi(x)\} = \Phi(p)$

Also, we have,

$$L\{\varphi'(x)\} = pL\{\varphi(x)\} - \varphi(0)$$
$$L\{\varphi''(x)\} = p^2L\{\varphi(x)\} - p\varphi(0) - \varphi'(0)$$

From the given conditions, $\varphi(0) = \varphi'(0) = 0$,

the above two equations will become,

$$L\{\varphi'(x)\} = pL\{\varphi(x)\}$$
$$= p\Phi(p)$$
$$L\{\varphi''(x)\} = p^2L\{\varphi(x)\}$$
$$= p^2\Phi(p)$$

Consider the given equation and apply the Laplace Transformation on both sides,

$L\{\varphi''(x)\} - 2L\{\varphi'(x)\} + L\{\varphi(x)\} + 2L\{\int_0^x cos(x-\ t)\varphi''(t)dt + \quad 2L\{\int_0^x \sin(x - t)\varphi'(t)dt\} = L\{cosx\}$

$L\{\varphi''(x)\} - 2L\{\varphi'(x)\} + L\{\varphi(x)\} + \ 2L\{cosx * \varphi''(x)\} + 2L\{sinx * \varphi'(x)\} = L\{cosx\}$

[Property-8]

$L\{\varphi''(x)\} - 2L\{\varphi'(x)\} + L\{\varphi(x)\} + \ 2L\{cosx\}L\{\varphi''(x)\} + 2L\{sinx\}L\{\varphi'(x)\} = L\{cosx\}$

Now, substitute the above values in this equation, we get,

$$p^2\Phi(p) - 2\,p\Phi(p) + \Phi(p) + 2\frac{p}{1+p^2}[p^2\Phi(p)] + 2\frac{1}{1+p^2}[p\Phi(p)] = \frac{p}{1+p^2}$$

$$\Phi(p)\left[p^2-2p+1+\frac{2p^3}{1+p^2}+\frac{2p}{1+p^2}\right]=\frac{p}{1+p^2}$$

$$\Phi(p)\left[\frac{p^4+2p^2+1}{1+p^2}\right]=\frac{p}{1+p^2}$$

$$\Phi(p)=\frac{p}{p^4+2p^2+1}$$

$$L\{\varphi(x)\}=\frac{p}{p^4+2p^2+1}$$

$$L\{\varphi(x)\}=\frac{p}{(p^2+1)^2}$$

$$\varphi(x)=L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}$$

$$\varphi(x)=\frac{xsinx}{2} \qquad \text{[Property-25]}$$

Hence, this is the required solution for the given integro-differential equation.

5.3.4. Example: Solve the integro-differential equation,

$\varphi''(x)+\varphi(x)+\int_0^x \sinh(x-t)\,\varphi(t)dt+\int_0^x \cosh(x-t)\,\varphi'(t)dt = coshx;\;\; \varphi(0)=-1,$ $\varphi'(0)=1,$

by using the Laplace Transformation.

Solution: Consider the given integro-differential equation,

$\varphi''(x)+\varphi(x)+\int_0^x \sinh(x-t)\,\varphi(t)dt+\int_0^x \cosh(x-t)\,\varphi'(t)dt = coshx;\; \varphi(0) = -1, \varphi'(0)=1$

Let us take the Laplace Transformation defined as, $L\{\varphi(x)\}=\Phi(p)$

Also, we have

$$L\{\varphi'(x)\}=pL\{\varphi(x)\}-\varphi(0)$$

$$L\{\varphi''(x)\}=p^2L\{\varphi(x)\}-p\varphi(0)-\varphi'(0)$$

From the given conditions, $\varphi(0)=-1, \varphi'(0)=1,$

the above two equations will become,

$$L\{\varphi'(x)\}=pL\{\varphi(x)\}+1$$
$$=p\Phi(p)+1$$
$$L\{\varphi''(x)\}=p^2L\{\varphi(x)\}+p-1$$
$$=p^2\Phi(p)+p-1$$

Consider the given equation and apply the Laplace transformation on both sides,

$L\{\varphi''(x)\}+L\{\varphi(x)\}+L\left\{\int_0^x \sinh(x-t)\,\varphi(t)dt\right\}+L\left\{\int_0^x \cosh(x-t)\,\varphi'(t)dt\right\} = L\{coshx\}$

$$L\{\varphi''(x)\} + L\{\varphi(x)\} + L\{\sinh(x) * \varphi(x)\} + L\{\cosh(x) * \varphi'(x)\}$$
$$= L\{coshx\}$$

[Property-8]

$$L\{\varphi''(x)\} + L\{\varphi(x)\} + L\{\sinh(x)\}\, L\{\varphi(x)\} + L\{\cosh(x)\}\, L\{\varphi'(x)\} = L\{coshx\}$$

$$p^2\Phi(p) + p - 1 + \Phi(p) + \frac{1}{p^2-1}\Phi(p) + \frac{p}{p^2-1}[p\Phi(p) + 1] = \frac{p}{p^2-1}$$

$$p^2\Phi(p) + p - 1 + \Phi(p) + \frac{\Phi(p)}{p^2-1} + \frac{p^2\Phi(p)}{p^2-1} + \frac{p}{p^2-1} = \frac{p}{p^2-1}$$

$$\Phi(p)\left[p^2 + 1 + \frac{1}{p^2-1} + \frac{p^2}{p^2-1}\right] = 1 - p$$

$$\Phi(p)\left[\frac{p^4+p^2}{p^2-1}\right] = 1 - p$$

$$\Phi(p) = \frac{(1-p)(p^2-1)}{(p^4+p^2)}$$

$$\Phi(p) = \frac{-p^3+p^2+p-1}{p^2(p^2+1)}$$

By taking partial fractions, we get,

$$\Phi(p) = \frac{1}{p} - \frac{1}{p^2} - \frac{2p}{p^2+1} + \frac{2}{p^2+1}$$

$$L\{\varphi(x)\} = \frac{1}{p} - \frac{1}{p^2} - \frac{2p}{p^2+1} + \frac{2}{p^2+1}$$

$$\varphi(x) = L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{1}{p^2}\right\} - L^{-1}\left\{\frac{2p}{p^2+1}\right\} + L^{-1}\left\{\frac{2}{p^2+1}\right\}$$

$$\varphi(x) = 1 - x - 2cosx + 2sinx$$

[Properties-18, 21, 20, 19]

$$\varphi(x) = 1 - x + 2(sinx - cosx)$$

Hence, this is the required solution for the given integro-differential equation.

5.3.5. Applications:

(1) Scientists and engineers encounter integro-differential equations through their research work in heat and mass diffusion processes, electric circuit problems, neutron diffusion, and biological species coexisting with increasing and decreasing rates of generation.

(2) The integro-differential equations in electro-magnetic theory, dispersive waves, and ocean circulations are enormous.

(3) These equations can be found in physics, biology, and engineering applications as well as in the advanced literature on integral equations.

5.4 SUMMARY:

In this lesson, we have discussed the integro-differential equations and their classification. The method of solving integro-differential equations using the Laplace transform was discussed in detail. In this regard, we recall some essential properties of the Laplace and inverse Laplace transforms. Certain examples and self-assessment problems related to integro-differential equations were discussed for a better understanding of the reader.

5.5 TECHNICAL TERMS:

Integro-differential equations: If the derivatives of the function are involved, it is called an Integro-differential equation.

$\varphi^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt,$

where, $\varphi^{(n)}(x) = \frac{d^n\varphi}{dx^n},$ and $K(x,t)$ be the kernel.

Kernel: If the kernel $K(x,t)$ of the integral equation is defined as a function of the difference (x,t), i.e.,$K(x,t) = K(x-t)$ where K is a certain function of one variable.

Laplace Transform: A function φ in terms of x and its Laplace transformation will be a function Φ in terms of p i.e., $L\{\varphi(x)\} = \Phi(p)$ (or) $\varphi(x) \risingdotseq \Phi(p)$.

i.e.,$L\{\varphi(x)\} = \int_0^\infty e^{-p}\ \varphi(x)dx = \Phi(p)$

5.6 SELF-ASSESSMENT QUESTIONS:

Exercise (5.1): Solve the following integro-differential equations:

1. $\varphi''(x) + \int_0^x e^{2(x-t)}\varphi'(t)dt = e^{2x};\quad \varphi(0) = 0, \varphi'(0) = 1$
2. $\varphi''(x) + 2\varphi'(x) - 2\int_0^x \sin(x-t)\,\varphi'(t)dt = cosx;\ \varphi(0) = \varphi'(0) = 0$
3. $\varphi''(x) + \varphi(x) + \int_0^x \sinh(x-t)\,\varphi(t)dt + \int_0^x \cosh(x-t)\,\varphi'(t)dt =$ $coshx;\ \varphi(0) = \varphi'(0) = 0$

Solutions to Exercise (5.1):

(1) $\varphi(x) = e^x - 1$

(2) $\varphi(x) = 1 - e^{-x} - xe^{-x}$

(3) $\varphi(x) = 1 - cosx$

5.7 SUGGESTED READINGS:

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LESSON- 6

VOLTERRA INTEGRAL EQUATION WITH LIMITS $(x, +\infty)$

OBJECTIVES:

- To learn about the Volterra integral equation with limits $(x, +\infty)$
- To discuss the Volterra integral equations of the first kind

STRUCTURE:

6.1. Introduction

6.2. Volterra Integral Equation

6.3. Volterra Integral Equation with Limits $(x, +\infty)$

6.4. Volterra Integral Equations of the First Kind

6.5 Summary

6.6 Technical Terms

6.7 Self-Assessment Questions

6.8 Suggested Readings

6.1. INTRODUCTION:

This lesson deals with the Volterra integral equations and their solution techniques. The principal investigators of the theory of integral equations are Vito Volterra (1860–1940) and Ivar Fredholm (1866–1927), together with David Hilbert (1862–1943) and Erhard Schmidt (1876–1959). Volterra was the first to recognize the importance of the theory and study it systematically.

6.2. VOLTERRA INTEGRAL EQUATION:

The most standard form of Volterra Linear Integral equations is of the form

$$\alpha(x)\varphi(x) = F(x) + \lambda \int_a^x K(x,t)\varphi(t)dt$$

where the limits of integration are the function of x and the unknown function $\varphi(x)$ appears linearly under the integral sign.

If the function $\alpha(x) = 1$, then the above equation simply becomes

$$\varphi(x) = F(x) + \lambda \int_a^x K(x,t)\varphi(t)dt$$

and this equation is known as the Volterra integral equation of the second kind;

whereas if $\alpha(x) = 0$, then the equation becomes

$$F(x) + \lambda \int_a^x K(x,t)\varphi(t)dt = 0,$$

which is known as the Volterra integral equation of the first kind.

6.3. VOLTERRA INTEGRAL EQUATIONS WITH LIMITS $(x, +\infty)$:

Integral equations of the form,

$$\varphi(x) = f(x) + \int_x^\infty K(x,t)\varphi(t)dt \qquad (1)$$

which arise in a number of problems in physics can also be solved by means of the Laplace transformation. For this purpose, we establish the convolution theorem for the expressions.

$$\int_x^\infty K(x,t)\varphi(t)dt \qquad (2)$$

It is known that for the Fourier transformation,

$$\mathcal{F}\{\int_{-\infty}^{+\infty} g(x-t)\psi(t)dt\} = \sqrt{2\pi}G(\lambda)\Psi(\lambda) \qquad (3)$$

where $G(\lambda), \Psi(\lambda)$ are the Fourier transforms of the functions $g(x)$ and $\psi(x)$ respectively. Put $g(x) = K_-(x)$, i.e.,

$$g(x) = \begin{cases} 0, & x > 0 \\ K(x), & x < 0 \end{cases}$$

$$\psi(x) = \varphi_+(x) = \begin{cases} \varphi(x), & x > 0 \\ 0, & x < 0 \end{cases} \qquad (4)$$

Then (3) can be rewritten as

$$\mathcal{F}\{\int_x^{+\infty} K(x-t)\varphi(t)dt\} = \sqrt{2\pi}\widetilde{K}_-(\lambda)_{\mathcal{F}}\widetilde{\Phi}_+(\lambda)_{\mathcal{L}} \qquad (5)$$

(here and henceforward, the subscripts $\mathcal{F}$ or $\mathcal{L}$ will mean that the Fourier transform or the Laplace transform of the function is taken).

To pass from the Fourier transform to the Laplace transform, observe that

$$F_{\mathcal{L}}(p) = \sqrt{2\pi}[F_+(ip)]_{\mathcal{F}} \qquad (6)$$

Hence, from (5) and (6) we get

$$\mathcal{L}\{\int_x^\infty K(x-t)\varphi(t)dt\} = \sqrt{2\pi}\left[\widetilde{K}_-(ip)\right]_{\mathcal{F}}[\Phi_+(p)]_{\mathcal{L}} \qquad (7)$$

We now express $\left[\sqrt{2\pi}\widetilde{K}_-(ip)\right]_{\mathcal{F}}$ in terms of the Laplace transform:

$$\left[\sqrt{2\pi}\widetilde{K}_-(ip)\right]_{\mathcal{F}} = \int_{-\infty}^0 K(x)e^{-px}dx = \int_0^\infty K(-x)e^{px}dx$$

Putting $K(-x) = \mathcal{K}(x)$, we get

$$\left[\sqrt{2\pi}\widetilde{K}_-(ip)\right]_{\mathcal{F}} = \widetilde{\mathcal{K}}_{\mathcal{L}}(-p) = \int_0^\infty K(-x)e^{px}dx$$

And so

$$\mathcal{L}\{\int_x^\infty K(x-t)\varphi(t)dt\} = \widetilde{\mathcal{K}}_{\mathcal{L}}(-p)\Phi_{\mathcal{L}}(p) \tag{8}$$

Let us now return to the integral equation (1). Taking the Laplace transform of both sides of (1), we obtain

$$\Phi(p) = F(p) + \widetilde{\mathcal{K}}(-p)\Phi(p) \tag{9}$$

(dropping the subscript $\mathcal{L}$) or

$$\Phi(p) = \frac{F(p)}{1-\widetilde{\mathcal{K}}(-p)} \qquad (\widetilde{\mathcal{K}}(-p) \neq 1) \tag{10}$$

where, $\widetilde{\mathcal{K}}(-p) = \int_0^\infty K(-x)e^{px}dx$ (11)

The function, $\varphi(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(p)}{1-\widetilde{\mathcal{K}}(-p)} e^{px}dp$ (12)

is a particular solution of the integral equation (1). It must be stressed that the solution (9) or (12) is meaningful only if the domains of analyticity of $\widetilde{\mathcal{K}}(-p)$ and $F(p)$ overlap.

6.3.1. Note:

(i) $\mathcal{L}\{f(x)\} = F(p)$

(ii) $\mathcal{L}\{\int_x^\infty K(x-t)\varphi(t)dt\} = \left(\int_0^\infty K(-x)e^{px}dx\right)\Phi(p)$

6.3.2. Note: Cauchy integral formula for residues can be expressed as,

$\int f(p)dp = 2\pi i$(sum of all residues)

$\frac{1}{2\pi i}\int f(p)dp$ = sum of all residues

6.3.3. Note: If $f(p)$ has a pole of order k at $p = p_0$ then

$$\text{Res}[f,p_0] = \frac{1}{(k-1)!}\lim_{p\to p_0}\frac{d^{k-1}}{dp^{k-1}}[(p-p_0)^k f(p)]$$

(i) If $f(p)$ has a pole of order 1, i.e., $k = 1$ at $p = p_0$ then,

$$\text{Res}[f,p_0] = \lim_{p\to p_0}(p-p_0)f(p)$$

(ii) If $f(p)$ has a pole of order 2, i.e., $k = 2$ at $p = p_0$ then,

$$\text{Res}[f,p_0] = \lim_{p\to p_0}\frac{d}{dp}[(p-p_0)^2 f(p)]$$

6.3.4. Example: Solve the integral equation, $\varphi(x) = x + \int_x^\infty e^{2(x-t)}\varphi(t)dt$.

Solution: Consider the given integral equation, $\varphi(x) = x + \int_x^\infty e^{2(x-t)}\varphi(t)dt$.

Compare the above equation with the general form, $\varphi(x) = f(x) + \int_x^\infty K(x,t)\varphi(t)dt$

Then we have, $f(x) = x,\ K(x,t) = e^{2(x-t)}$.

So, $K(x) = e^{2x}$, then $K(-x) = e^{-2x}$

Now, apply the Laplace transform on both sides of the given equation,

$$\mathcal{L}\{\varphi(x)\} = \mathcal{L}\{x\} + \mathcal{L}\{\int_x^\infty e^{2(x-t)}\varphi(t)dt\}$$

$$\Phi(p) = \frac{1}{p^2} + \left(\int_0^\infty e^{-2x}e^{px}dx\right)\Phi(p)$$

[∵We know that, $\mathcal{L}\{\varphi(x)\} = \Phi(p)$, and from Property-12 in 5.2.9 also from (ii) in 6.3.1]

$$\Phi(p) = \frac{1}{p^2} + \frac{1}{2-p}\Phi(p)$$

$$\Phi(p)\left[1 - \frac{1}{2-p}\right] = \frac{1}{p^2}$$

$$\Phi(p) = \frac{p-2}{p^2(p-1)}$$

From (12) in section 6.3, we get,

$$\varphi(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{p-2}{p^2(p-1)} e^{px}dp \qquad \text{(A)}$$

The above integral can be evaluated by using the Cauchy integral formula with residues. The integrand function has a double pole at $p = 0$ and a simple pole at $p = 1$.

Now, calculate the residue for the pole $p = 0$ of order 2,

$$= \lim_{p\to 0}\frac{d}{dp}\left[p^2\frac{p-2}{p^2(p-1)}e^{px}\right]$$

$$= \lim_{p\to 0}\frac{d}{dp}\left[\frac{p-2}{p-1}e^{px}\right]$$

$$= \lim_{p\to 0}\left[\frac{p-2}{p-1}e^{px}x + e^{px}\left(\frac{1}{(p-1)^2}\right)\right]$$

$$= 2x + 1$$

Also, calculate the residue for the pole $p = 1$ of order 1,

$$= \lim_{p\to 1}\left[(p-1)\ \frac{p-2}{p^2(p-1)}e^{px}\right]$$

$$= \lim_{p\to 1}\left[\frac{p-2}{p^2}e^{px}\right]$$

$$= -e^x$$

Now, using 6.3.2 in equation (A), we get,

$$\varphi(x) = 2x + 1 - e^x$$

Hence, this is the required solution for the given integral equation.

6.3.5. Example: Solve the integral equation, $\varphi(x) = e^{-x} + \int_x^\infty e^{x-t}\varphi(t)dt$

Solution: Consider the given integral equation, $\varphi(x) = e^{-x} + \int_x^\infty e^{x-t}\varphi(t)dt$

Compare the above equation with the general form, $\varphi(x) = f(x) + \int_x^\infty K(x,t)\varphi(t)dt$

Here, $f(x) = e^{-x}$, $K(x,t) = e^{x-t}$

So, $K(x) = e^{x}$, then $K(-x) = e^{-x}$

Now, apply the Laplace Transform on both sides of the given equation,

$$\mathcal{L}\{\varphi(x)\} = \mathcal{L}\{e^{-x}\} + \mathcal{L}\{\int_x^\infty e^{x-t}\varphi(t)dt\}$$

$$\Phi(p) = \frac{1}{p+1} + \left(\int_0^\infty e^{-x}e^{px}dx\right)\Phi(p)$$

[We know that, $\mathcal{L}\{\varphi(x)\} = \Phi(p)$, and from property-13 in 5.2.9 also from (ii) in 6.3.1]

$$\Phi(p) = \frac{1}{p+1} + \left(\int_0^\infty e^{-x}e^{px}dx\right)\Phi(p)$$

$$\Phi(p) = \frac{1}{p+1} - \frac{1}{p-1}\Phi(p)$$

$$\Phi(p)\left[1 + \frac{1}{p-1}\right] = \frac{1}{p+1}$$

$$\Phi(p) = \frac{p-1}{p(p+1)}$$

From (12) in section 6.3, we get,

$$\varphi(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{p-1}{p(p+1)} e^{px}dp \qquad \text{(A)}$$

The above integral can be evaluated by using the Cauchy integral formula with residues. The integrand function has the simple poles at $p = 0$ and $p = -1$.

Now, calculate the residue for the pole $p = 0$ of order 1.

$$= \lim_{p\to 0}\left[p\,\frac{p-1}{p(p+1)}e^{px}\right]$$

$$= \lim_{p\to 0}\left[\frac{p-1}{p+1}e^{px}\right]$$

$$= -1$$

Also, calculate the residue for the pole $p = -1$ of order 1,

$$= \lim_{p\to -1}\left[(p+1)\frac{p-1}{p(p+1)}e^{px}\right]$$

$$= \lim_{p\to -1}\left[\frac{p-1}{p}e^{px}\right]$$

$$= 2e^{-x}$$

Now, using 6.3.2 in equation (A), we get,

$$\varphi(x) = -1 + 2e^{-x}$$

Hence, this is the required solution for the given integral equation.

6.3.6. Example: Solve the integral equation, $\varphi(x) = 1 + \int_x^\infty e^{\alpha(x-t)}\varphi(t)dt, \quad (\alpha > 0)$

Solution: Consider the given integral equation, $\varphi(x) = 1 + \int_x^\infty e^{\alpha(x-t)}\varphi(t)dt$

Compare the above equation with the general form, $\varphi(x) = f(x) + \int_x^\infty K(x,t)\varphi(t)dt$

Here, $f(x) = 1, \; K(x,t) = e^{\alpha(x-t)}$
So, $K(x) = e^{\alpha x}$, then $K(-x) = e^{-\alpha x}$

Now, apply the Laplace transform on both sides of the given equation,

$$\mathcal{L}\{\varphi(x)\} = \mathcal{L}\{1\} + \mathcal{L}\{\int_x^\infty e^{\alpha(x-t)}\varphi(t)dt\}$$

$$\Phi(p) = \frac{1}{p} + \left(\int_0^\infty e^{-\alpha x}e^{px}dx\right)\Phi(p)$$

[∵We know that, $\mathcal{L}\{\varphi(x)\} = \Phi(p)$, and from Property-9 in 5.2.9 also from (ii) in 6.3.1]

$$\Phi(p) = \frac{1}{p} - \frac{1}{p-\alpha}\Phi(p)$$

$$\Phi(p)\left[1 + \frac{1}{p-\alpha}\right] = \frac{1}{p}$$

$$\Phi(p) = \frac{p-\alpha}{p[p-(\alpha-1)]}$$

From (12) in section 6.3, we get,

$$\varphi(x) = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{p-\alpha}{p[p-(\alpha-1)]}e^{px}dp \qquad \text{(A)}$$

The above integral can be evaluated by using the Cauchy integral formula with residues. The integrand function has the simple poles at $p = 0$ and $p = \alpha - 1$.

Now, calculate the residue for the pole $p = 0$ of order 1,

$$= \lim_{p\to 0}\left[p\,\frac{p-\alpha}{p[p-(\alpha-1)]}e^{px}\right]$$

$$= \lim_{p\to 0}\left[\frac{p-\alpha}{p-(\alpha-1)}e^{px}\right]$$

$$= \frac{\alpha}{\alpha-1}$$

Also, calculate the residue for pole $p = \alpha - 1$ of order 1,

$$= \lim_{p\to\alpha-1}\left[[p-(\alpha-1)]\frac{p-\alpha}{p[p-(\alpha-1)]}e^{px}\right]$$

$$= \lim_{p \to \alpha-1} \left[\frac{p-\alpha}{p} e^{px}\right]$$

$$= \frac{-1}{\alpha-1} e^{(\alpha-1)x}$$

Now, using 6.3.2 in equation (A), we get,

$$\varphi(x) = \frac{\alpha}{\alpha-1} - \frac{1}{\alpha-1} e^{(\alpha-1)x}$$

Hence, this is the required solution for the given integral equation.

6.4. VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND:

Suppose we have a Volterra integral equation of the first kind,

$$\int_0^x K(x,t)\varphi(t)dt = f(x), \ \ f(0) = 0 \tag{1}$$

where, $\varphi(x)$ is the unknown function.

Suppose that $K(x,t), \frac{\partial K(x,t)}{\partial x}, f(x)$ and $f'(x)$ are continuous for $0 \le x \le a, 0 \le t \le x$. Differentiating both sides of equation (1) with respect to x, we obtain

$$K(x,x)\varphi(x) + \int_0^x \frac{\partial K(x,t)}{\partial x} \varphi(t)dt = f'(x) \tag{2}$$

The above equation can be obtained by using the Leibniz rule in 5.2.6.

Any continuous solution $\varphi(x)$ of equation (1), for $0 \le x \le a$,obviously satisfies equation (2) as well. Conversely, any continuous solution of equation (2), for $0 \le x \le a$, satisfies equation (1) too.

If $K(x,x)$ does not vanish at any point of the basic interval [0, a], then equation (2) can be rewritten as,

$$\varphi(x) = \frac{f'(x)}{K(x,x)} - \int_0^x \frac{K'_x(x,t)}{K(x,x)} \varphi(t)dt \tag{3}$$

which means it reduces to a Volterra-type integral equation of the second kind.

If $K(x,x) \equiv 0$, then it is sometimes useful to differentiate (2) once again with respect to x and so on.

6.4.1. Example: Solve the integral equation, $\int_0^x \cos(x-t)\,\varphi(t)dt = x$.

Solution: Consider the given integral equation, $\int_0^x \cos(x-t)\,\varphi(t)dt = x$.
Compare the given equation with the general form, $\int_0^x K(x,t)\varphi(t)dt = f(x)$.

Here the functions $f(x) = x,\ \ K(x,t) = \cos(x-t)$, satisfy the conditions of continuity and differentiability.

Differentiating both sides of the given equation with respect to x by using Leibniz's rule, we get

$$-\int_0^x \sin(x-t)\,\varphi(t)dt + \cos(x-x)\,\varphi(x)(1) - \cos(x-0)\,\varphi(0)(0) = 1$$

$$-\int_0^x \sin(x-t)\,\varphi(t)dt + \cos(0)\,\varphi(x) = 1$$

$$-\int_0^x \sin\,(x-t)\,\varphi(t)dt + \varphi(x) = 1$$

$$\varphi(x) = 1 + \int_0^x \sin\,(x-t)\,\varphi(t)dt$$

The above equation is an integral equation of the second kind of the convolution type.

We find its solution by applying the Laplace transformation,

$$L\{\varphi(x)\} = L\{1\} + L\left\{\int_0^x \sin\,(x-t)\,\varphi(t)dt\right\}$$

$$L\{\varphi(x)\} = L\{1\} + L\{sinx * \varphi(x)\} \quad \text{[Property-8]}$$

$$L\{\varphi(x)\} = L\{1\} + L\{sinx\} * L\{\varphi(x)\}$$

$$\Phi(p) = \frac{1}{p} + \frac{1}{p^2+1}\Phi(p) \quad \text{[Properties-9, 10]}$$

$$\Phi(p)\left[1 - \frac{1}{p^2+1}\right] = \frac{1}{p}$$

$$\Phi(p) = \frac{p^2+1}{p^3}$$

$$\Phi(p) = \frac{1}{p} + \frac{1}{p^3}$$

$$L\{\varphi(x)\} = \frac{1}{p} + \frac{1}{p^3}$$

$$\varphi(x) = L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$\varphi(x) = 1 + \frac{x^2}{2} \quad \text{[Properties-18,28]}$$

Hence, it is the required solution.

6.4.2. Example: Solve the integral equation, $\int_0^x \mathrm{e}^{x-t}\varphi(t)dt = sinx$.

Solution: Consider the given integral equation, $\int_0^x \mathrm{e}^{x-t}\varphi(t)dt = sinx$.

Compare the given equation with the general form, $\int_0^x K(x,t)\varphi(t)dt = f(x)$.

Here the functions $f(x) = sinx,\ \ K(x,t) = \mathrm{e}^{x-t}$ satisfy the conditions of continuity and differentiability.

Differentiating both sides of the given equation with respect to x, by using Leibniz's rule, we get,

$$\int_0^x \mathrm{e}^{x-t}\,\varphi(t)dt + \mathrm{e}^{x-x}\varphi(x)(1) - \mathrm{e}^{x-0}\varphi(0)(0) = cosx$$

$$\int_0^x \mathrm{e}^{x-t}\,\varphi(t)dt + \varphi(x) = cosx$$

$$\varphi(x) = cosx - \int_0^x e^{x-t}\,\varphi(t)dt$$

The above equation is an integral equation of the second kind of the convolution type.
We find the solution by applying the Laplace transformation,

$$L\{\varphi(x)\} = L\{cosx\} - L\{\int_0^x e^{x-t}\,\varphi(t)dt\}$$

$$L\{\varphi(x)\} = L\{cosx\} - L\{e^x * \varphi(x)\} \qquad \text{[Property-8]}$$

$$L\{\varphi(x)\} = L\{cosx\} - L\{e^x\} * L\{\varphi(x)\}$$

$$\Phi(p) = \frac{p}{p^2+1} - \frac{1}{p-1}\Phi(p) \qquad \text{[Properties-11, 13]}$$

$$\Phi(p)\left[1 + \frac{1}{p-1}\right] = \frac{p}{p^2+1}$$

$$\Phi(p) = \frac{p-1}{p^2+1}$$

$$\Phi(p) = \frac{p}{p^2+1} - \frac{1}{p^2+1}$$

$$L\{\varphi(x)\} = \frac{p}{p^2+1} - \frac{1}{p^2+1}$$

$$\varphi(x) = L^{-1}\left\{\frac{p}{p^2+1}\right\} - L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$\varphi(x) = cosx - sinx \qquad \text{[Properties 20, 19]}$$

Hence, it is the required solution.

6.4.3. Example: Solve the integral equation, $\int_0^x (1 - x^2 + t^2)\varphi(t)dt = \frac{x^2}{2}$

Solution: Consider the given integral equation, $\int_0^x (1 - x^2 + t^2)\varphi(t)dt = \frac{x^2}{2}$

Compare the given equation with the general form, $\int_0^x K(x,t)\varphi(t)dt = f(x)$.

Here the functions $f(x) = \frac{x^2}{2}$, $K(x,t) = (1 - x^2 + t^2)$ satisfy the conditions of continuity and differentiability.

Differentiating both sides of the given equation with respect to x by using Leibniz's rule, we get $\int_0^x \frac{\partial}{\partial x}(1 - x^2 + t^2)\,\varphi(t)dt + (1 - x^2 + x^2)\varphi(x)(1) - (1 - x^2 + 0^2)\varphi(0)(0) = \frac{1}{2}(2x)$

$$\int_0^x -2x\,\varphi(t)dt + \varphi(x) = x$$

$$\varphi(x) = x + 2\int_0^x x\,\varphi(t)dt$$

The above equation is an integral equation of the second kind of the convolution type.

By the method of successive approximation (3.1 in Lesson 3), we have,

$$f(x) = x,\ K(x,t) = x,\ \lambda = 2$$

So, $\varphi_0(x) = f(x)$

$= x$

$\varphi_1(x) = \int_0^x K(x,t)\varphi_0(t)dt$

$= \int_0^x xtdt$

$= x\left(\frac{t^2}{2}\right)_0^x$

$= x\left(\frac{x^2}{2} - 0\right)$

$= \frac{x^3}{2}$

$\varphi_2(x) = \int_0^x K(x,t)\varphi_1(t)dt$

$= \int_0^x x\frac{t^3}{2}dt$

$= \frac{x}{2}\left(\frac{t^4}{4}\right)_0^x$

$= \frac{x}{2}\left(\frac{x^4}{4} - 0\right)$

$= \frac{x^5}{8}$

and so on...

Now, $\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\varphi_n(x)$

$= x + 2\frac{x^3}{2} + 2^2\frac{x^5}{8} + \cdots$

$= x + x^3 + \frac{x^5}{2} + \cdots$

$= x(1 + x^2 + x^4 + \cdots)$

$= x(1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \cdots)$

$= xe^{x^2}$ $\left[\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$

Hence, it is the required solution.

6.4.4. Example: Solve the integral equation, $\int_0^x (2 + x^2 - t^2)\varphi(t)dt = x^2$

Solution: Consider the given integral equation, $\int_0^x (2 + x^2 - t^2)\varphi(t)dt = x^2$

Compare the given equation with the general form,

$$\int_0^x K(x,t)\varphi(t)dt = f(x)$$

Here the functions $f(x) = x^2,\ \ K(x,t) = (2 + x^2 - \mathrm{t}^2)$ satisfy the conditions of continuity and differentiability.

Differentiating both sides of the given equation with respect to x by using Leibniz's rule, we get,

$$\int_0^x \frac{\partial}{\partial \mathrm{x}}(2 + x^2 - \mathrm{t}^2)\,\varphi(t)dt + (2 + x^2 - x^2)\varphi(x)(1) - (2 + x^2 - 0)\varphi(0)(0) = 2x$$

$$\int_0^x 2x\,\varphi(t)dt + 2\varphi(x) - 0 = 2x$$

$$\int_0^x 2x\,\varphi(t)dt + 2\varphi(x) = 2x$$

$$\int_0^x x\,\varphi(t)dt + \varphi(x) = x$$

$$\varphi(x) = x - \int_0^x x\,\varphi(t)dt$$

The above equation is an integral equation of the second kind of the convolution type.

By the method of successive approximation (3.1 in Lesson 3), we have,

$$f(x) = x,\ K(x,t) = x,\ \lambda = -1$$

Now, $\varphi_0(x) = f(x)$

$= x$

$$\varphi_1(x) = \int_0^x K(x,t)\varphi_0(t)dt$$

$$= \int_0^x xtdt$$

$$= x\left(\frac{t^2}{2}\right)_0^x$$

$$= \frac{x}{2}(x^2 - 0)$$

$$= \frac{x^3}{2}$$

$$\varphi_2(x) = \int_0^x K(x,t)\varphi_1(t)dt$$

$$= \int_0^x x\left(\frac{t^3}{2}\right)dt$$

$$= \frac{x}{2}\left(\frac{t^4}{4}\right)_0^x$$

$$= \frac{x}{8}(x^4 - 0)$$

$$= \frac{x^5}{8}$$

and so on...

Now, $\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots + \lambda^n\varphi_n(x)$

$$= x + (-1)\left(\frac{x^3}{2}\right) + (-1)^2\frac{x^5}{8} + \cdots$$

$$= x - \frac{x^3}{2} + \frac{x^5}{8} + \cdots$$

$$= x\left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right)$$

$$= x\left(1 - \frac{\left(\frac{x^2}{2}\right)}{1!} + \frac{\left(\frac{x^2}{2}\right)^2}{2!} + \cdots\right)$$

$$\varphi(x) = xe^{\frac{-x^2}{2}} \qquad \left[\begin{array}{l} \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \end{array}\right]$$

Hence, it is the required solution.

6.5 SUMMARY:

In this section, we explore the Volterra integral equation with limits $(x, +\infty)$. Here, we discuss the procedure for finding solutions to Volterra integral equations of the first kind in detail. A few examples and self-assessment questions are provided to enhance the reader's understanding.

6.6 TECHNICAL TERMS:

Volterra Integral Equation of First Kind:
The most standard form of Volterra linear integral equations is,

$$\alpha(x)\varphi(x) = F(x) + \lambda \int_a^x K(x,t)\varphi(t)dt,$$

if $\alpha = 0$, then $F(x) + \lambda \int_a^x K(x,t)\varphi(t)dt = 0$ is the Volterra integral equation of the first kind.

Residue: If $f(p)$ has a pole of order k at $p = p_0$ then,

$$\text{Res}[f, p_0] = \frac{1}{(k-1)!} \lim_{p \to p_0} \frac{d^{k-1}}{dp^{k-1}} [(p - p_0)^k f(p)]$$

6.7 SELF-ASSESSMENT QUESTIONS:

Exercise (6.1): Solve the integral equations:

(1) $\varphi(x) = e^{-x} + \int_x^\infty \varphi(t)dt$

(2) $\varphi(x) = cosx + \int_x^\infty e^{(x-t)}\varphi(t)dt$

(3) $\int_0^x 3^{x-t}\varphi(t)dt = x$

(4) $\int_0^x \mathrm{a}^{x-t}\varphi(t)dt = f(x),\ f(0) = 0$

(5) $\int_0^x sin(x - \mathrm{t})\varphi(t)dt = \mathrm{e}^{\frac{x^2}{2}} - 1$

Solutions to Exercise (6.1):

(1) $\varphi(x) = (1 - x)e^{-x}$

(2) $\varphi(x) = cosx - sinx$

(3) $\varphi(x) = 1 - xln3$

(4) $\varphi(x) = f'(x) - f(x)lna$

(5) $\varphi(x) = e^{\frac{x^2}{2}}(x^2 + 2) - 1$

6.8 SUGGESTED READINGS:

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LESSON- 7

EULER INTEGRALS AND ABEL'S INTEGRAL EQUATIONS

OBJECTIVES:

- To learn about Euler integrals
- To discuss Abel's problem and Abel's integral equation
- To learn the concept of generalizations of Abel's integral equation

STRUCTURE:

7.1 Euler Integrals

7.2 Abel's Problem and Abel's Integral Equation

7.3 Generalizations of Abel's Integral Equation

7.4 Summary

7.5 Technical Terms

7.6 Self-Assessment Questions

7.7 Suggested Readings

7.1. Euler Integrals:

The gamma function or Euler's integral of the second kind is the function $\Gamma(x)$ defined by the equality,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (1)$$

where, x is any complex number, $\mathrm{Re}(x) > 0$.

For $x = 1$, we get

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \quad (2)$$

Integrating by parts, we obtain from (1)

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-t} t^{x} dt$$
$$= \frac{\Gamma(x+1)}{x} (3)$$

This equation expresses the basic property of a gamma function

$$\Gamma(x+1) = x\Gamma(x) \quad (4)$$

Using (2), we get

$$\Gamma(2) = \Gamma(1+1) = 1.\Gamma(1) = 1,$$
$$\Gamma(3) = \Gamma(2+1) = 2.\Gamma(2) = 2!$$
$$\Gamma(4) = \Gamma(3+1) = 3.\Gamma(3) = 3!$$

and generally for positive integral n,

$$\Gamma(n) = (n-1)! \quad (5)$$

We know that,

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

Putting $x = t^{\frac{1}{2}}$ here, we obtain,

$$\int_0^\infty e^{-t}t^{\frac{1}{2}-1}\,dt = \sqrt{\pi}$$

Taking into account expression (1) for the gamma function, we can write this equation as,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Also, we can obtain

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1\times 3}{2^2}\sqrt{\pi}$$

and so on.

Generally, it will readily be seen that the following equality holds:

$$\Gamma\left(\mathrm{n}+\frac{1}{2}\right) = \frac{1\times 3\times 5\ldots(2n-1)}{2^n}\sqrt{\pi}$$

(n is a positive integer).

Knowing the value of the gamma function for some value of the argument, we can compute, from (3), the value of the function for an argument diminished by unity. For example,

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

For this reason,

$$\Gamma\left(\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\frac{1}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\frac{1}{2}} = \sqrt{\pi}$$

Acting in a similar fashion, we find

$$\Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{-1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-1}{2}} = -2\sqrt{\pi}$$
$$\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{\frac{-3}{2}} = \frac{\Gamma\left(\frac{-1}{2}\right)}{\frac{-3}{2}} = \frac{4}{3}\sqrt{\pi}$$
$$\Gamma\left(\frac{-5}{2}\right) = \frac{\Gamma\left(\frac{-5}{2}+1\right)}{\frac{-5}{2}} = \frac{\Gamma\left(\frac{-3}{2}\right)}{\frac{-5}{2}} = \frac{-8}{15}\sqrt{\pi}$$

and so on.

It is easy to verify that,

$$\Gamma(0) = \Gamma(-1) = \cdots = \Gamma(-n) = \cdots = \infty.$$

Above we defined $\Gamma(x)$ for $\mathrm{Re}(x) > 0$. The indicated method for computing $\Gamma(x)$ extends this function into the left half-plane, where $\Gamma(x)$ is defined everywhere except at the points $x = -n$ (n is a positive integer and 0).

Note also the following relations:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\pi^{\frac{1}{2}}\Gamma(2x)$$

and generally,

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-n}\;\Gamma(nx)$$

(Gauss-Legendre Multiplication theorem).

The gamma function was represented by Weierstrass by means of the equation,

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}}\right\} \quad (6)$$

where,

$$\gamma = \lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}-\ln m\right) = 0.57721\ldots$$

is Euler's constant. From (6), it is evident that the function $\Gamma(z)$ is analytic everywhere except at $z = 0,\ z = -1,\ z = -2, \ldots$, where it has simple poles.

The following is Euler's formula, which is obtained from (6);

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right\}$$

It holds everywhere except at $z = 0,\ z = -1,\ z = -2, \ldots$

7.1.1. Example: Show that for $Re(z) > 0$

$$\Gamma(z) = \int_0^1 \left(ln\frac{1}{x}\right)^{z-1} dx$$

Solution:
Let us define gamma function as,

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt \qquad (1)$$

Now, we have to show that,

$$\Gamma(z) = \int_0^1 \left(ln\frac{1}{x}\right)^{z-1} dx$$

Consider RHS of above equation,

$$\text{RHS} = \int_0^1 \left(ln\frac{1}{x}\right)^{z-1} dx (2)$$

Let us assume that, $log\frac{1}{x} = t$

$$\frac{1}{x} = e^t$$

$$x = e^{-t}$$
$$dx = -e^{-t}dt$$

The limits of integration will change as under $t = \infty$ as $x = 0$ and $t = 0$ as $x = 1$.

Now, substitute all the above values in equation (2), which gives

$$\begin{aligned}
\text{RHS} &= \int_0^1 \left(ln\frac{1}{x}\right)^{z-1} dx \\
&= \int_\infty^0 t^{z-1}(-e^{-t})dt \\
&= \int_0^\infty e^{-t}t^{z-1}dt \left(\because \int_a^b f(x)dx = -\int_b^a f(x)dx\right) \\
&= \Gamma(z)(\because \text{ equation (1)}) \\
&= \text{LHS}
\end{aligned}$$

Hence, $\Gamma(z) = \int_0^1 \left(ln\frac{1}{x}\right)^{z-1} dx$.

7.1.2. Note:

We introduce Euler's integral of the first kind $B(p,q)$, so-called beta function:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (Re\, p > 0, Re\, q > 0)$$

The following equality holds (it establishes a relationship between the Euler integrals of the first and second kinds):

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Also, we have some alternate definitions of the beta function,

$$B(p,q) = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}dx$$

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} sin^{2p-1}\theta\, cos^{2q-1}\theta d\theta$$

$$B(p,q) = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx$$

Also, we have,

$$\int_0^{\frac{\pi}{2}} sin^p\theta\, cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

7.1.3. Example: Show that, $B(p,q) = B(q,p)$

Solution: Let us define the beta function as,

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (Re\, p > 0, Re\, q > 0) \quad (1)$$

Consider the LHS of the given expression, i.e.,

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (\because \text{ Equation } (1))$$
$$=\int_0^1 (1-x)^{p-1}\big(1-(1-x)\big)^{q-1}dx$$
$$\left(\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$
$$=\int_0^1 (1-x)^{p-1}x^{q-1}dx$$
$$=\int_0^1 x^{q-1}(1-x)^{p-1}dx$$
$$=B(q,p)$$

Hence, $B(p,q)=B(q,p)$.

7.1.4. Example: Show that,

$$B(p,q)=B(p+1,q)+B(p,q+1)$$

Solution: Let us define the beta function as,

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{1}$$

Now, consider the RHS of the given expression as,

$$B(p+1,q)+B(p,q+1)=\frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)}+\frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}$$
$$(\because \text{ Equation } (1))$$
$$=\frac{p\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)}+\frac{\Gamma(p)q\Gamma(q)}{(p+q)\Gamma(p+q)}$$
$$=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\left(\frac{p}{p+q}+\frac{q}{p+q}\right)$$
$$=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\left(\frac{p+q}{p+q}\right)$$
$$=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$
$$=B(p,q) \qquad (\because \text{ Equation } (1))$$

Hence, $B(p,q)=B(p+1,q)+B(p,q+1)$.

7.1.5. Example: Show that,

$$\int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1}\,dx=2^{p+q-1}B(p,q)$$

Solution: Consider the LHS of the given equation,

$$\text{LHS}=\int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1}\,dx \tag{1}$$

Let us assume that, $1+x=2t$ then $dx=2dt$

The limits of integration will change as under $t=0$ as $x=-1$ and $t=1$ as $x=1$.

Now, substitute all the above values in equation (1),

$$\text{LHS}=\int_0^1 (2t)^{p-1}\big(1-(2t-1)\big)^{q-1}\,2dt$$
$$=\int_0^1 (2t)^{p-1}(2-2t)^{q-1}\,2dt$$
$$=\int_0^1 2^{p-1}\,t^{p-1}2^{q-1}(1-t)^{q-1}2dt$$

$$= 2^{p+q-1}\int_0^1 t^{p-1}(1-t)^{q-1}dt$$

$$= 2^{p+q-1}B(p,q)\left(\because B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx\ \right)$$

= RHS

Hence, $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}B(p,q)$.

7.1.6. Example: Evaluate the integral,

$$I = \int_0^{\frac{\pi}{2}} cos^{m-1}x\ sin^{n-1}x\ dx \quad (Re\ m > 0, Re\ n > 0).$$

Solution: Consider the given integral,

$$I = \int_0^{\frac{\pi}{2}} cos^{m-1}x\ sin^{n-1}x\ dx$$

Now, compare with the general form,

$$\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\ d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Here, we have

$$p = n-1, q = m-1$$

Hence, given the integral becomes,

$$I = \frac{\Gamma\left(\frac{n-1+1}{2}\right)\Gamma\left(\frac{m-1+1}{2}\right)}{2\Gamma\left(\frac{n-1+m-1+2}{2}\right)}$$

$$I = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)}{2\Gamma\left(\frac{m+n}{2}\right)}$$

Hence, it is the required solution.

7.2. ABEL'S PROBLEM AND ABEL'S INTEGRAL EQUATION:

7.2.1. Definition:

An integral equation is called a singular integral equation if one or both limits of integration become infinite, or if the kernel $K(x,t)$ of the equation becomes infinite at one or more points in the interval of integration. To be specific, the integral equation of the first kind,

$$f(x) = \lambda\int_a^b K(x,t)\varphi(t)dt \qquad (1)$$

or the integral equation of the second kind

$$\varphi(x) = f(x) + \lambda\int_a^b K(x,t)\varphi(t)dt \qquad (2)$$

is called singular if a, b or both limits of integration are infinite. Equation (1) or (2) is also called a singular equation if the kernel $K(x,t)$ becomes infinite at one or more points in the domain of integration.

Consider the following examples,

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt \qquad (3)$$

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}\varphi(t)dt \qquad (4)$$

The above integral equations (3) and (4) are called Abel's problem and generalized Abel's integral equation, respectively.

One of the simplest forms of singular integral equations, which arises in mechanics, is Abel's integral equation

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} \varphi(t)dt\,, \quad 0 < \alpha < 1 \qquad (5)$$

where $\varphi(t)$ is an unknown function to be determined and $f(x)$ is a known function.

Multiplying (5) both sides by $(u-x)^{-(1-\alpha)}$ and integrate with regard to x from 0 to u, we have

$$\int_0^u \frac{f(x)}{(u-x)^{-(1-\alpha)}} dx = \int_{x=0}^u \frac{dx}{(u-x)^{1-\alpha}} \int_{t=0}^x \frac{\varphi(t)}{(x-t)^\alpha} dt$$

By changing the order of integration, we have

$$\int_0^u \frac{f(x)}{(u-x)^{-(1-\alpha)}} dx = \int_{t=0}^u \varphi(t)dt \int_{x=t}^u \frac{dx}{(u-x)^{1-\alpha}(x-t)^\alpha} \qquad (6)$$

Consider,

$$z = \frac{u-x}{u-t}$$
$$dx = -(u-t)dz$$

Then,

$$\int_{x=t}^u \frac{dx}{(u-x)^{1-\alpha}(x-t)^\alpha}$$
$$= -\int_{z=1}^0 [z(u-t)]^{\alpha-1}(u-t)^{-\alpha}(1-z)^{-\alpha}(u-t)dz$$
$$= \int_{z=0}^1 z^{\alpha-1}(1-z)^{-\alpha}dz$$
$$= \frac{\pi}{sin\alpha\pi}$$
$$= B(\alpha, 1-\alpha) \qquad (7)$$

From relations (6) and (7), we have

$$\int_0^u \frac{f(x)}{(u-x)^{-(1-\alpha)}} dx = \frac{\pi}{sin\alpha\pi} \int_{t=0}^u \varphi(t)dt \qquad (8)$$

Differentiating the relation (8) with regard to u and then changing u by t, we obtain

$$\varphi(t) = \frac{sin\alpha\pi}{\pi} \frac{d}{dt}\left[\int_0^t f(x)(t-x)^{\alpha-1}dx\right],$$

which determines the solution of the given equation (5).

7.2.2. Relation between Laplace Transformation and Gamma function:

(1) $L\{x^n\} = \frac{\Gamma(n+1)}{p^{n+1}} = \frac{n!}{p^{n+1}}$, if $n\epsilon z^+$

(2) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{x^n}{\Gamma(n+1)}$

(3) $L\left\{\frac{1}{\sqrt{x}}\right\} = L\left\{x^{\frac{-1}{2}}\right\} = \frac{\Gamma(1-\frac{1}{2})}{p^{1-\frac{1}{2}}} = \sqrt{\frac{\pi}{p}}$

(4) $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = \frac{1}{\sqrt{\pi}\sqrt{x}}$

7.2.3. Example: Solve the integral equation,

$$\int_0^x \frac{\varphi(t)dt}{(x-t)^\alpha} = x^n \; (0 < \alpha < 1)$$

Solution: Consider the given integral equation,

$$\int_0^x \frac{\varphi(t)dt}{(x-t)^\alpha} = x^n$$

Now, applying the Laplace transform on both sides, we get,

$$L\left\{\int_0^x \frac{\varphi(t)dt}{(x-t)^\alpha}\right\} = L\{x^n\}$$

$$L\left\{\frac{1}{x^\alpha} * \varphi(x)\right\} = L\{x^n\} \qquad (\because \text{Property-8 in 5.2.9 })$$

$$L\left\{\frac{1}{x^\alpha}\right\} * L\{\varphi(x)\} = L\{x^n\}$$

$$L\{x^{-\alpha}\}L\{\varphi(x)\} = L\{x^n\}$$

$$\frac{\Gamma(-\alpha+1)}{p^{-\alpha+1}} L\{\varphi(x)\} = \frac{\Gamma(\text{n}+1)}{p^{n+1}} \qquad \left(\because L\{x^n\} = \frac{\Gamma(\text{n}+1)}{p^{n+1}}\right)$$

$$L\{\varphi(x)\}\Gamma(-\alpha+1) = \frac{\Gamma(\text{n}+1)}{p^n p} p^{-\alpha} p$$

$$L\{\varphi(x)\}\Gamma(-\alpha+1) = \frac{\Gamma(\text{n}+1)}{p^{n+\alpha}}$$

$$L\{\varphi(x)\} = \frac{\Gamma(\text{n}+1)}{\Gamma(1-\alpha)} \frac{1}{p^{n+\alpha}}$$

$$\varphi(x) = \frac{\Gamma(\text{n}+1)}{\Gamma(1-\alpha)} L^{-1}\left\{\frac{1}{p^{n+\alpha}}\right\}$$

$$\varphi(x) = \frac{\Gamma(\text{n}+1)}{\Gamma(1-\alpha)} L^{-1}\left\{\frac{1}{p^{n+\alpha-1+1}}\right\}$$

$$\varphi(x) = \frac{\Gamma(\text{n}+1)}{\Gamma(1-\alpha)} \frac{x^{n+\alpha-1}}{\Gamma(n+\alpha-1+1)}$$

$$\varphi(x) = \frac{\Gamma(\text{n}+1)}{\Gamma(1-\alpha)} \frac{x^{n+\alpha-1}}{\Gamma(n+\alpha)}$$

Hence, it is the required solution.

7.2.4. Example: Solve the integral equation,

$$\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = sinx$$

Solution: Consider the given integral equation,

$$\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = sinx$$

Now, applying the Laplace transform on both sides, we get,

$$L\left\{\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}}\right\} = L\{sinx\}$$

$$L\left\{\frac{1}{\sqrt{x}} * \varphi(x)\right\} = \frac{1}{1+p^2} \qquad (\because \text{Property-8 in 5.2.9})$$

$$L\left\{\frac{1}{\sqrt{x}}\right\} L\{\varphi(x)\} = \frac{1}{1+p^2}$$

$$\sqrt{\frac{\pi}{p}} L\{\varphi(x)\} = \frac{1}{1+p^2} \qquad (\because 7.2.2)$$

$$L\{\varphi(x)\} = \sqrt{\frac{p}{\pi}} \frac{1}{1+p^2}$$

$$L\{\varphi(x)\} = \frac{1}{\sqrt{\pi}} \frac{\sqrt{p}}{1+p^2}$$

$$L\{\varphi(x)\} = \frac{1}{\sqrt{\pi}} \left(\frac{p}{\sqrt{p}(1+p^2)}\right)$$

$$\varphi(x) = \frac{1}{\sqrt{\pi}} L^{-1}\left\{\frac{1}{\sqrt{p}} \frac{p}{1+p^2}\right\}$$
$$\varphi(x) = \frac{1}{\sqrt{\pi}} \left(L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} L^{-1}\left\{\frac{p}{1+p^2}\right\}\right)$$
$$\varphi(x) = \frac{1}{\sqrt{\pi}} \left(\frac{1}{\sqrt{\pi}\sqrt{x}} cosx\right) \quad (\because \text{7.2.2 and 5.2.10 })$$
$$\varphi(x) = \frac{1}{\pi} \left(\frac{1}{\sqrt{x}} cosx\right)$$
$$\varphi(x) = \frac{1}{\pi} \left(\int_0^x \frac{cost}{\sqrt{x-t}} dt\right)$$

Hence, it is the required solution.

7.2.5. Example: Solve the integral equation,

$$\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = x^{\frac{1}{2}}$$

Solution: Consider the given integral equation,

$$\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = x^{\frac{1}{2}}$$

Now, applying the Laplace transform on both sides, we get,

$$L\left\{\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}}\right\} = L\left\{x^{\frac{1}{2}}\right\}$$
$$L\left\{\frac{1}{\sqrt{x}} * \varphi(x)\right\} = L\left\{x^{\frac{1}{2}}\right\}$$
$$L\left\{\frac{1}{\sqrt{x}}\right\} L\{\varphi(x)\} = L\left\{x^{\frac{1}{2}}\right\}$$
$$\sqrt{\frac{\pi}{p}} L\{\varphi(x)\} = \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}$$
$$\left(\because 7.2.2 \; and \; L\left\{x^{\frac{1}{2}}\right\} = \frac{\Gamma\left(\frac{3}{2}\right)}{p^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\right)$$
$$L\{\varphi(x)\} = \frac{1}{2p}$$
$$\varphi(x) = \frac{1}{2} L^{-1}\left\{\frac{1}{p}\right\}$$
$$\varphi(x) = \frac{1}{2} \left(\because \; L^{-1}\left\{\frac{1}{p}\right\} = 1\right)$$

Hence, it is the required solution.

7.3. GENERALIZATIONS OF ABEL'S INTEGRAL EQUATION:

Consider the integral equation,

$$\int_0^x (x-t)^\beta \varphi(t)dt = x^\lambda \tag{1}$$

$(\lambda \geq 0, \;\; \beta > -1 \; is \; real \,)$,
which in a sense is a further generalization of Abel's equation (5) in 7.2.1.

Multiply both sides of equation (1) by $(z-x)^\mu$ $(\mu > -1)$ and integrate with respect to x from 0 to z:

$$\int_0^z (z-x)^\mu \left(\int_0^x (x-t)^\beta \varphi(t)dt\right)dx = \int_0^z x^\lambda (z-x)^\mu dx \tag{2}$$

Putting $x = \rho z$ in the integral on the right side of (2), we obtain

$$\int_0^z x^\lambda \, (z-x)^\mu dx = z^{\lambda+\mu+1} \int_0^1 \rho^\lambda (1-\rho)^\mu d\rho$$
$$= z^{\lambda+\mu+1} B(\lambda+1, \mu+1)$$

$$= z^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}\quad (\lambda+\mu+1>\lambda\geq 0) \qquad (3)$$

Changing the order of integration on the left side of (2), we get,

$$\int_0^z\left(\int_0^x (z-x)^\mu (x-t)^\beta \varphi(t)dt\right)dx =$$
$$\int_0^z\left(\int_t^z (z-x)^\mu (x-t)^\beta dx\right)\varphi(t)dt \qquad (4)$$

In the inner integral on the right of (4) put

$$x = t+\rho(z-t)$$

Then,

$$\int_t^z (z-x)^\mu (x-t)^\beta dx = (z-t)^{\mu+\beta+1}\int_0^1 \rho^\beta (1-\rho)^\mu d\rho$$

$$\int_t^z (z-x)^\mu (x-t)^\beta dx = (z-t)^{\mu+\beta+1}\mathrm{B}(\beta+1,\mu+1)$$

$$\int_t^z (z-x)^\mu (x-t)^\beta dx = (z-t)^{\mu+\beta+1}\frac{\Gamma(\beta+1)\Gamma(\mu+1)}{\Gamma(\beta+\mu+2)} \qquad (5)$$

Taking into account (3), (4), (5), we obtain from (2),

$$\int_0^z\left((z-t)^{\mu+\beta+1}\frac{\Gamma(\beta+1)\Gamma(\mu+1)}{\Gamma(\beta+\mu+2)}\right)\varphi(t)dt = z^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}$$

$$\frac{\Gamma(\beta+1)\Gamma(\mu+1)}{\Gamma(\beta+\mu+2)}\int_0^z (z-t)^{\mu+\beta+1}\ \varphi(t)dt = z^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)\Gamma(\mu+1)}{\Gamma(\lambda+\mu+2)}$$

$$\frac{\Gamma(\beta+1)}{\Gamma(\beta+\mu+2)}\int_0^z (z-t)^{\mu+\beta+1}\ \varphi(t)dt = z^{\lambda+\mu+1}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+2)} \qquad (6)$$

Choose μ so that $\mu+\beta+1=n$ (a non-negative integer). Then from (6) we will have

$$\frac{\Gamma(\beta+1)}{\Gamma(n+1)}\int_0^z (z-t)^n\ \varphi(t)dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+n-\beta+1)}z^{\lambda+n-\beta}$$

$$\int_0^z \frac{(z-t)^n}{n!}\varphi(t)dt = \frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\Gamma(\lambda+n-\beta+1)}z^{\lambda+n-\beta}$$

$$(\because \Gamma(n+1)=n!) \qquad (7)$$

Differentiating both sides of (7) $(n+1)$ times with respect to z, we obtain

$$\varphi(z) = \frac{\Gamma(\lambda+1)(\lambda+n-\beta)(\lambda+n-\beta-1)\ldots(\lambda-\beta)}{\Gamma(\beta+1)(\lambda+n-\beta+1)}z^{\lambda-\beta-1} \qquad (8)$$

or for $\lambda-\beta+k\neq 0\ (k=0,1,\ldots,n)$

$$\varphi(z) = \frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\Gamma(\lambda-\beta)}z^{\lambda-\beta-1} \qquad (9)$$

This is the solution of the integral equation (1).

7.3.1. Example: Solve the integral equation,

$$\int_0^x (x-t)\varphi(t)dt = x^2$$

Solution: Consider the given integral equation,

$$\int_0^x (x-t)\varphi(t)dt = x^2$$

Compare the given integral equation with,

$$\int_0^x (x-t)^\beta\varphi(t)dt = x^\lambda$$

Here, $\lambda=2, \beta=1$.

Since, $\lambda-\beta+k\neq 0\ (k=0,1,\ 2,3,\ldots,n)$, it follows from the general formula that,

$$\varphi(z) = \frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\Gamma(\lambda-\beta)}z^{\lambda-\beta-1}$$

$$\varphi(x) = \frac{\Gamma(2+1)}{\Gamma(1+1)\Gamma(2-1)} x^{2-1-1}$$

$$\varphi(x) = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} x^0$$

$$\varphi(x) = \frac{2!}{(1)(1)}(1)$$

$$\varphi(x) = 2$$

Hence, it is the required solution.

7.3.2. Example: Solve the integral equation,

$$\int_0^x (x-t)^{\frac{1}{3}} \varphi(t)dt = x^{\frac{4}{3}} - x^2$$

Solution: Consider the given integral equation,

$$\int_0^x (x-t)^{\frac{1}{3}} \varphi(t)dt = x^{\frac{4}{3}} - x^2$$

Compare the given integral equation with,

$$\int_0^x (x-t)^{\beta} \varphi(t)dt = x^{\lambda}$$

Here, $\lambda_1 = \frac{4}{3}, \lambda_2 = 2, \beta = \frac{1}{3}$.

Since, $\lambda_1 - \beta + k \neq 0,\ \lambda_2 - \beta + k \neq 0 (k = 0, 1,\ 2, 3, \dots, n)$,

it follows from the general formula that,

$$\varphi(z) = \frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\Gamma(\lambda-\beta)} z^{\lambda-\beta-1}$$

For this problem we have two λ values, so the solution will be,

$$\varphi(z) = \frac{\Gamma(\lambda_1+1)}{\Gamma(\beta+1)\Gamma(\lambda_1-\beta)} z^{\lambda_1-\beta-1} - \frac{\Gamma(\lambda_2+1)}{\Gamma(\beta+1)\Gamma(\lambda_2-\beta)} z^{\lambda_2-\beta-1}$$

$$\varphi(x) = \frac{\Gamma\left(\frac{4}{3}+1\right)}{\Gamma\left(\frac{1}{3}+1\right)\Gamma\left(\frac{4}{3}-\frac{1}{3}\right)} x^{\frac{4}{3}-\frac{1}{3}-1} - \frac{\Gamma(2+1)}{\Gamma\left(\frac{1}{3}+1\right)\Gamma\left(2-\frac{1}{3}\right)} x^{2-\frac{1}{3}-1}$$

$$\varphi(x) = \frac{\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma(1)} x^0 - \frac{\Gamma(3)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)} x^{\frac{2}{3}}$$

$$\varphi(x) = \frac{\frac{4}{3}\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}\right)} - \frac{2!}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)} x^{\frac{2}{3}}$$

$$\varphi(x) = \frac{4}{3} - \frac{2}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{5}{3}\right)} x^{\frac{2}{3}}$$

Hence, it is the required solution.

7.3.3. Example: Solve the integral equation,

$$\int_0^x (x-t)^{\frac{1}{2}} \varphi(t)dt = \pi x.$$

Solution: Consider the given integral equation,

$$\int_0^x (x-t)^{\frac{1}{2}} \varphi(t)dt = \pi x$$

Compare the given integral equation with,

$$\int_0^x (x-t)^{\beta} \varphi(t)dt = x^{\lambda}$$

Here, $\lambda = 1, \beta = \frac{1}{2}$.

Since, $\lambda - \beta + k \neq 0\ (k = 0, 1,\ 2, 3, \dots, n)$, it follows from the general formula that,

$$\varphi(z) = \frac{\Gamma(\lambda+1)}{\Gamma(\beta+1)\Gamma(\lambda-\beta)} z^{\lambda-\beta-1}$$

$$\varphi(x) = \pi\left[\frac{\Gamma(1+1)}{\Gamma\left(\frac{1}{2}+1\right)\Gamma(1-\frac{1}{2})} x^{1-\frac{1}{2}-1}\right]$$

$$\varphi(x) = \pi\left[\frac{\Gamma(1+1)}{\Gamma\left(\frac{1}{2}+1\right)\Gamma(1-\frac{1}{2})} x^{1-\frac{1}{2}-1}\right]$$

$$\varphi(x) = \pi\left[\frac{\Gamma(1+1)}{\Gamma\left(\frac{1}{2}+1\right)\Gamma(1-\frac{1}{2})} x^{1-\frac{1}{2}-1}\right]$$

$$\varphi(x) = \pi\left[\frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)\Gamma(\frac{1}{2})} \frac{1}{\sqrt{x}}\right]$$

$$\varphi(x) = \pi\left[\frac{1}{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}} \frac{1}{\sqrt{x}}\right]$$

$$\left(\because \Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}\right)$$

$$\varphi(x) = \pi\left[\frac{2}{\pi\sqrt{x}}\right]$$

$$\varphi(x) = \frac{2}{\sqrt{x}}$$

Hence, it is the required solution.

7.4 SUMMARY:

This lesson provides a detailed description of Euler integrals. Later deals with the concepts of Abel's problem and Abel's integral equation. Also, discussed the generalizations of Abel's integral equation. Certain examples of all these concepts are provided to enhance the reader's understanding.

7.5 TECHNICAL TERMS:

Euler Integrals:

(1) Euler's integral of the second kind $\Gamma(x)$ is called gamma function and is defined by,

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-\mathrm{t}}\mathrm{t}^{x-1}\mathrm{dt}$$

where, x is any complex number, $\mathrm{Re}(x) > 0$.

(2) Euler's integral of the first kind $B(p,q)$ is called the beta function and is defined by,

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx \quad (Re\, p > 0, Re\, q > 0)$$

Relation between Euler integrals of the first and second kind:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Abel's Integral Equation: The simplest form of singular integral equations, which arises in mechanics, is Abel's integral equation.

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha}\varphi(t)dt, \quad 0 < \alpha < 1$$

where $\varphi(t)$ is an unknown function to be determined and $f(x)$ is a known function.

Generalizations of Abel's Integral Equation: The integral equation,

$$\int_0^x (x-t)^{\beta}\varphi(t)dt = x^{\lambda} \quad (\lambda \geq 0,\ \beta > -1\ is\ real\ )$$

which, in a sense, is a generalization of the Abel integral equation.

7.6 SELF ASSESSMENT QUESTIONS:

Exercise (7.1): Solve the following:

(1) Show that, $\Gamma'(1) = -\gamma$

(2) Show that, $\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = 2.ln2$

(3) Prove that, $\Gamma(z) = \lim_{n\to\infty} \frac{1.2\ldots\ldots(n-1)}{z(z+1)\ldots\ldots(z+n-1)} n^z$

(4) Show that, $B(p+1,q) = \frac{p}{q}B(p,q+1)$

Exercise(7.2): Solve the integral equations:

(1) $\int_0^x (x-t)^{\frac{1}{4}}\varphi(t)dt = x + x^2$

(2) $\int_0^x (x-t)^2\varphi(t)dt = x^3$

(3) $\frac{1}{2}\int_0^x (x-t)^2\varphi(t)dt = cosx - 1 + \frac{x^2}{2}$

(4) $\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = e^x$

Solutions to Exercise(7.2):

(1) $\varphi(x) = \frac{1}{\Gamma\left(\frac{5}{4}\right)}\left[\frac{1}{\Gamma\left(\frac{3}{4}\right)x^{\frac{1}{4}}} + \frac{2x^{\frac{3}{4}}}{\Gamma\left(\frac{7}{4}\right)}\right]$

(2) $\varphi(x) = 3$

(3) $\varphi(x) = sinx$

(4) $\varphi(x) = \frac{1}{\pi}\left(\frac{1}{\sqrt{x}} + e^x \int_0^x e^{-t} t^{\frac{-1}{2}} dt\right)$

7.7 SUGGESTED READINGS:

1. Shanti Swarup, Integral equations, Krishna Prakashan Pvt Ltd, Meerut, 2003.
2. M Krasnov, A Kiselev, G Makarenko, Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971.
3. M Rahman, Integral equations and their applications, WIT Press, Southampton, Boston, 2007.
4. Erwin Kreyszig, Advanced Engineering Mathematics, Wiley International Publication, 2010.

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LESSON- 8

VOLTERRA INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE

OBJECTIVES:

- To learn about Volterra integral equations of the first kind of the convolution type
- To know the necessary conditions for the existence of a solution of an integral equation
- To discuss Volterra integral equations of the first kind with a logarithmic kernel
- To learn about the non-linear Volterra integral equations with
- convolution type

STRUCTURE:

8.1. Volterra Integral Equations of the First Kind of the Convolution Type

8.2. Necessary Condition for the Existence of a Solution of an Integral Equation

8.3. Volterra Integral Equations of the First Kind with Logarithmic Kernel

8.4. Non-Linear Volterra Integral Equations with Convolution Type

8.5 Summary

8.6 Technical Terms

8.7 Self-Assessment Questions

8.8 Suggested Readings

8.1. VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND OF THE CONVOLUTION TYPE:

8.1.1. Definition: An integral equation of the first kind,

$$\int_0^x K(x-t)\varphi(t)dt = f(x) \quad (1)$$

whose kernel $K(x,t)$ is dependent solely on the difference $(x-t)$ of arguments, will be called an integral equation of the first kind of the convolution type. This class of equations includes, for instance, the generalized Abel's equation.
Let us consider a problem that leads to a Volterra integral equation of the convolution type.

8.1.2. Problem: A shop buys and sells a variety of commodities. It is assumed that,

(1) buying and selling are continuous processes, and purchased goods are put on sale at once;
(2) the shop acquires each new lot of any type of goods in quantities which it can sell in a time interval T, the same for all purchases;
(3) each new lot of goods is sold uniformly over time T, the total cost of which is unity.

It is required to find the law $\varphi(t)$ by which it should make purchases so that the cost of goods on hand should be constant.

Solution: Let the cost of the original goods on hand at time t be equal to $K(t)$ where,

$$K(t) = \begin{cases} 1 - \frac{t}{T}, & t \le T \\ 0, & t > T \end{cases}$$

Let us suppose that in the time interval between τ and $\tau + d\tau$

goods are bought, amounting to the sum of $\varphi(\tau)d\tau$. This reserve diminishes (due to sales) in such a manner that the cost of the remaining goods at the time $t > \tau$ is equal to $K(t-\tau)\varphi(\tau)d\tau$. Therefore, the cost of the unsold part of goods acquired via purchases will, at any time t be equal to

$$\int_0^t K(t-\tau)\varphi(\tau)d\tau$$

Thus, $\varphi(t)$ should satisfy the integral equation

$$1 - K(t) = \int_0^t K(t-\tau)\varphi(\tau)d\tau$$

We have thus obtained a Volterra integral equation of the first kind of the convolution type.

Let $f(x)$ and $K(x)$ be original functions and let

$$f(x) \risingdotseq F(p), K(x) \risingdotseq \widetilde{K}(p), \varphi(x) \risingdotseq \Phi(p)$$

Taking the Laplace Transform of both sides of equation (1) and utilizing the convolution theorem, we will have

$$\widetilde{K}(p)\,\Phi(p) = F(p) \qquad (2)$$

$$\Phi(p) = \frac{F(p)}{\widetilde{K}(p)}\left(\widetilde{K}(p) \ne 0\right) \qquad (3)$$

The original function $\varphi(x)$ for the function $\Phi(p)$ defined by equation (3) will be a solution of the integral equation (1).

8.1.3. Example: Solve the integral equation,

$$\int_0^x e^{x-t}\varphi(t)dt = x$$

Solution: Consider the given integral equation,

$$\int_0^x e^{x-t}\varphi(t)dt = x$$

Compare the given integral equation with the general form,

$$\int_0^x K(x-t)\varphi(t)dt = f(x)$$

Here, $f(x) = x,\ K(x-t) = e^{x-t}$

So, $K(x) = e^x$

Now, take the Laplace transform on both sides of the given equation, we get,

$$L\left\{\int_0^x e^{x-t}\varphi(t)dt\right\} = L\{x\}$$

$$L\{e^x * \varphi(x)\} = L\{x\} \qquad (\because \text{Properties -5.2.9})$$

$$L\{e^x\}L\{\varphi(x)\} = L\{x\}$$

$\frac{1}{p-1}\Phi(p) = \frac{1}{p^2}$ (∵ Properties -5.2.9)

$\Phi(p) = \frac{p-1}{p^2}$

$\Phi(p) = \frac{1}{p} - \frac{1}{p^2}$

$L\{\varphi(x)\} = \frac{1}{p} - \frac{1}{p^2}$

$\varphi(x) = 1 - x$ (∵ Properties -5.2.10)

Hence, it is the required solution

8.1.4. Example: Solve the integral equation,

$$\int_0^x \cos(x-t)\,\varphi(t)dt = sinx$$

Solution: Consider the given integral equation,

$$\int_0^x \cos(x-t)\varphi(t)dt = sinx$$

Compare the given integral equation with the general form,

$$\int_0^x K(x-t)\varphi(t)dt = f(x)$$

Here, $f(x) = sinx,\ K(x-t) = \cos(x-t)$

So, $K(x) = cosx$

Now, take the Laplace transform on both sides of the given equation, we get,

$$L\left\{\int_0^x \cos(x-t)\varphi(t)dt\right\} = L\{sinx\}$$

$$L\{cosx * \varphi(x)\} = L\{sinx\}$$

(∵ Properties -5.2.9)

$$L\{cosx\}L\{\varphi(x)\} = L\{sinx\}$$

$\frac{p}{1+p^2}\Phi(p) = \frac{1}{1+p^2}$ (∵ Properties -5.2.9)

$$\Phi(p) = \frac{1}{p}$$

$$L\{\varphi(x)\} = \frac{1}{p}$$

$\varphi(x) = 1$ (∵ Properties -5.2.10)

Hence, it is the required solution.

8.1.5. Example: Solve the integral equation,

$$\int_0^x (x-t)^{\frac{1}{2}}\varphi(t)dt = x^{\frac{5}{2}}$$

Solution: Consider the given integral equation,

$$\int_0^x (x-t)^{\frac{1}{2}}\varphi(t)dt = x^{\frac{5}{2}}$$

Compare the given integral equation with the general form,

$$\int_0^x K(x-t)\varphi(t)dt = f(x)$$

Here, $f(x) = x^{\frac{5}{2}},\ K(x-t) = (x-t)^{\frac{1}{2}}$

So, $K(x) = x^{\frac{1}{2}}$

Now, take the Laplace transform on both sides of the given equation, we get

$$L\left\{\int_0^x (x-t)^{\frac{1}{2}}\varphi(t)dt\right\} = L\left\{x^{\frac{5}{2}}\right\}$$

$$L\left\{x^{\frac{1}{2}} * \varphi(x)\right\} = L\left\{x^{\frac{5}{2}}\right\}$$

$$L\left\{x^{\frac{1}{2}}\right\}L\{\varphi(x)\} = L\left\{x^{\frac{5}{2}}\right\}$$

$$\frac{\Gamma\left(\frac{1}{2}+1\right)}{p^{\frac{1}{2}+1}}L\{\varphi(x)\} = \frac{\Gamma\left(\frac{5}{2}+1\right)}{p^{\frac{5}{2}+1}} \qquad (\because \text{ Properties -7.2.2})$$

$$\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{p^{\frac{3}{2}}}L\{\varphi(x)\} = \frac{\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}{p^{\frac{7}{2}}}$$

$$\frac{\frac{1}{2}\sqrt{\pi}}{p^{\frac{3}{2}}}L\{\varphi(x)\} = \frac{\frac{5}{2}\times\frac{3}{4}\sqrt{\pi}}{p^{\frac{7}{2}}}$$

$$L\{\varphi(x)\} = \frac{\frac{5}{2}\times\frac{3}{4}}{p^{\frac{7}{2}}} \times 2p^{\frac{3}{2}}$$

$$L\{\varphi(x)\} = \frac{15}{8p^{\frac{7}{2}}} \times 2p^{\frac{3}{2}}$$

$$L\{\varphi(x)\} = \frac{15}{4} \times \frac{p^{\frac{3}{2}}}{p^{\frac{7}{2}}}$$

$$L\{\varphi(x)\} = \frac{15}{4p^2}$$

$$\varphi(x) = \frac{15}{4}L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$\varphi(x) = \frac{15}{4}x \qquad (\because \text{ Properties -5.2.10})$$

Hence, it is the required solution.

8.1.6. Example: Solve the integral equation,

$$\int_0^x \cos(x-t)\,\varphi(t)dt = x + x^2$$

Solution: Consider the given integral equation,

$$\int_0^x \cos(x-t)\,\varphi(t)dt = x + x^2$$

Compare the given integral equation with the general form,

$$\int_0^x K(x-t)\varphi(t)dt = f(x)$$

Here, $f(x) = x + x^2,\ K(x-t) = \cos(x-t)$

So, $K(x) = cosx$

Now, take the Laplace transform on both sides of given equation, we get,

$$L\{\int_0^x \cos(x-t)\,\varphi(t)dt\} = L\{x + x^2\}$$

$$L\{cosx * \varphi(x)\} = L\{x + x^2\} \qquad (\because \text{Properties -5.2.9})$$

$$L\{cosx\}L\{\varphi(x)\} = L\{x\} + L\{x^2\}$$

$$\frac{p}{1+p^2}\Phi(p) = \frac{1}{p^2} + \frac{2}{p^3} \qquad (\because \text{Properties -5.2.9})$$

$$\Phi(p) = \left(\frac{1}{p^2} + \frac{2}{p^3}\right)\frac{1+p^2}{p}$$

$$\Phi(p) = \frac{1}{p^2}\left(\frac{1+p^2}{p}\right) + \frac{2}{p^3}\left(\frac{1+p^2}{p}\right)$$

$$\Phi(p) = \frac{1}{p^2}\left(\frac{1}{p} + p\right) + \frac{2}{p^3}\left(\frac{1}{p} + p\right)$$

$$\Phi(p) = \frac{1}{p^3} + \frac{1}{p} + \frac{2}{p^4} + \frac{2}{p^2}$$

$$L\{\varphi(x)\} = \frac{1}{p^3} + \frac{1}{p} + \frac{2}{p^4} + \frac{2}{p^2}$$

$$\varphi(x) = L^{-1}\left\{\frac{1}{p^3}\right\} + L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{2}{p^4}\right\} + L^{-1}\left\{\frac{2}{p^2}\right\}$$

$$\varphi(x) = \frac{x^2}{2} + 1 + 2\frac{x^3}{6} + 2x \qquad (\because \text{Properties -5.2.9})$$

$$\varphi(x) = 1 + 2x + \frac{x^2}{2} + \frac{x^3}{3}$$

Hence, it is the required solution.

8.1.7. Example: Solve the integral equation,

$$\int_0^x e^{2(x-t)}\varphi(t)dt = sinx$$

Solution: Consider the given integral equation,

$$\int_0^x e^{2(x-t)}\varphi(t)dt = sinx$$

Compare the given integral equation with the general form,

$$\int_0^x K(x-t)\varphi(t)dt = f(x)$$

Here, $f(x) = sinx,\ K(x-t) = e^{2(x-t)}$

So, $K(x) = e^{2x}$

Now, take the Laplace transform on both sides of the given equation, we get,

$$L\{\int_0^x e^{2(x-t)}\varphi(t)dt\} = L\{sinx\}$$

$$L\{e^{2x} * \varphi(x)\} = L\{sinx\} \qquad (\because \text{Properties -5.2.9})$$

$$L\{e^{2x}\}L\{\varphi(x)\} = L\{sinx\}$$

$$\frac{1}{p-2}\Phi(p) = \frac{1}{1+p^2} \qquad (\because \text{Properties -5.2.9})$$

$$\Phi(p) = \frac{p-2}{1+p^2}$$

$$\Phi(p) = \frac{p}{1+p^2} - \frac{2}{1+p^2}$$

$$L\{\varphi(x)\} = \frac{p}{1+p^2} - \frac{2}{1+p^2}$$

$$\varphi(x) = cosx - 2sinx \quad (\because \text{Properties -5.2.10})$$

Hence, it is the required solution.

8.2. NECESSARY CONDITION FOR THE EXISTENCE OF A SOLUTION OF AN INTEGRAL EQUATION:

A necessary condition for the existence of a continuous solution of an integral equation of the form,

$$\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\varphi(t)dt = f(x) \qquad (1)$$

consists in the function $f(x)$ having continuous derivatives up to the n^{th} order inclusive and in all its $n-1$ first derivatives vanishing for $x = 0$. This model equation (1) points to the necessity of matching the orders of vanishing of the kernel for $t = x$ and of the right side $f(x)$ for $x = 0$ (the right side must exceed the left by at least unity).

To find the solution $\varphi(x)$ of equation (1), apply the Laplace transform on both sides of the given equation. Then assume $\varphi(x)$ as $\delta(x)$, i.e.,

$$\varphi(x) = \delta(x).$$

This is made clear by direct verification if we take into account that the convolution of the δ-function and any other smooth function $g(x)$ is defined as,

$$g(x) * \delta(x) = g(x)$$

$$\delta^{(k)}(x) * g(x) = g^{(k)}(x) \quad (k = 1, 2, \dots)$$

Indeed, in our case $g(x) = K(x)$ and

$$\int_0^x K(x-t)\,\delta(t)dt = K(x).$$

8.2.1. Example: Solve the integral equation,

$$\int_0^x (x-t)\varphi(t)dt = x^2 + x - 1$$

Solution: Consider the given integral equation,

$$\int_0^x (x-t)\varphi(t)dt = x^2 + x - 1$$

Now, take the Laplace Transform on both sides of the given equation,

$$L\left\{\int_0^x (x-t)\varphi(t)dt\right\} = L\{x^2 + x - 1\}$$

$$L\{x * \varphi(x)\} = L\{x^2\} + L\{x\} - L\{1\}$$

(∵ Properties -5.2.9)

$$L\{x\}L\{\varphi(x)\} = L\{x^2\} + L\{x\} - L\{1\}$$

$$\frac{1}{p^2}\Phi(p) = \frac{2}{p^3} + \frac{1}{p^2} - \frac{1}{p}$$

$$\Phi(p) = \left(\frac{2}{p^3} + \frac{1}{p^2} - \frac{1}{p}\right)p^2$$

$$\Phi(p) = \frac{2}{p} + 1 - p$$

$$L\{\varphi(x)\} = \frac{2}{p} + 1 - p$$

$$\varphi(x) = L^{-1}\left\{\frac{2}{p} + 1 - p\right\}$$

$$\varphi(x) = L^{-1}\left\{\frac{2}{p}\right\} + L^{-1}\{1\} - L^{-1}\{p\}$$

$$\varphi(x) = 2L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\{1\} - L^{-1}\{p\}$$

$$\varphi(x) = 2(1) + \delta(x) - \delta'(x)$$

$$\left(\because\ L^{-1}\left\{\frac{1}{p}\right\} = 1,\ L^{-1}\{1\} = \delta(x),\ L^{-1}\{p\} = \delta'(x)\right)$$

$$\varphi(x) = 2 + \delta(x) - \delta'(x)$$

Hence, it is the required solution.

8.2.2. Example: Solve the integral equation,

$$\int_0^x (x-t)\varphi(t)dt = sinx$$

Solution: Consider the given integral equation,

$$\int_0^x (x-t)\varphi(t)dt = sinx$$

Now, take the Laplace Transform on both sides of the given equation,

$$L\left\{\int_0^x (x-t)\varphi(t)dt\right\} = L\{sinx\}$$

$$L\{x * \varphi(x)\} = L\{sinx\}$$

(∵ Properties -5.2.9)

$$L\{x\}L\{\varphi(x)\} = L\{sinx\}$$

$$\frac{1}{p^2}\Phi(p) = \frac{1}{1+p^2} \qquad (\because \text{ Properties -5.2.9})$$

$$\Phi(p) = \frac{p^2}{1+p^2}$$

$$\Phi(p) = \frac{p^2+1-1}{1+p^2}$$

$$\Phi(p) = 1 - \frac{1}{1+p^2}$$

$$L\{\varphi(x)\} = 1 - \frac{1}{1+p^2}$$

$$\varphi(x) = L^{-1}\left\{1 - \frac{1}{1+p^2}\right\}$$

$$\varphi(x) = L^{-1}\{1\} - L^{-1}\left\{\frac{1}{1+p^2}\right\}$$

$$\varphi(x) = \delta(x) - sinx$$

[$\because L^{-1}\{1\} = \delta(x)$, Properties-5.2.10]

Hence, it is the required solution.

8.2.3. Example: Solve the integral equation,

$$\int_0^x (x-t)^2 \varphi(t)dt = x^2 + x^3$$

Solution: Consider given integral equation,

$$\int_0^x (x-t)^2 \varphi(t)dt = x^2 + x^3$$

Now, take the Laplace Transform on both sides of given equation,

$$L\{x^2 * \varphi(x)\} = L\{x^2 + x^3\} \qquad (\because \text{Properties -5.2.9})$$

$$L\{x^2\}L\{\varphi(x)\} = L\{x^2\} + L\{x^3\}$$

$$\frac{2}{p^3}\Phi(p) = \frac{2}{p^3} + \frac{6}{p^4} \qquad (\because \text{Properties -5.2.9})$$

$$\Phi(p) = \frac{\left(\frac{2}{p^3}+\frac{6}{p^4}\right)}{\frac{2}{p^3}}$$

$$\Phi(p) = 1 + \frac{3}{p}$$

$$L\{\varphi(x)\} = 1 + \frac{3}{p}$$

$$\varphi(x) = L^{-1}\left\{1 + \frac{3}{p}\right\}$$

$$\varphi(x) = L^{-1}\{1\} + L^{-1}\left\{\frac{3}{p}\right\}$$

$$\varphi(x) = L^{-1}\{1\} + 3L^{-1}\left\{\frac{1}{p}\right\}$$

$$\varphi(x) = \delta(x) + 3$$

$\left(\because\ L^{-1}\{1\} = \delta(x), L^{-1}\left\{\frac{1}{p}\right\} = 1\ \right)$ Hence, it is the required solution.

8.2.4. Example: Solve the integral equation,

$$\int_0^x \sin(x-t)\,\varphi(t)dt = x + 1$$

Solution: Consider the given integral equation,

$$\int_0^x \sin\,(x-t)\varphi(t)dt = x + 1$$

Now, take the Laplace Transform on both sides of the given equation,

$$L\{\int_0^x \sin(x-t)\varphi(t)dt\} = L\{x+1\}$$

$$L\{sinx * \varphi(x)\} = L\{x+1\} \quad (\because \text{ Properties -5.2.9})$$

$$L\{sinx\}L\{\varphi(x)\} = L\{x\} + L\{1\}$$

$$\frac{1}{1+p^2}\Phi(p) = \frac{1}{p^2} + \frac{1}{p} \quad (\because \text{ Properties -5.2.9})$$

$$\Phi(p) = \left(\frac{1}{p^2} + \frac{1}{p}\right)(1+p^2)$$

$$\Phi(p) = \frac{1+p^2}{p^2} + \frac{1+p^2}{p}$$

$$\Phi(p) = \frac{1}{p^2} + 1 + \frac{1}{p} + p$$

$$L\{\varphi(x)\} = \frac{1}{p^2} + 1 + \frac{1}{p} + p$$

$$\varphi(x) = L^{-1}\left\{\frac{1}{p^2}\right\} + L^{-1}\{1\} + L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\{p\}$$

$$\varphi(x) = x + \delta(x) + 1 + \delta'(x)$$

[$\because$ Properties -5.2.10, $L^{-1}\{1\} = \delta(x), L^{-1}\{p\} = \delta'(x)$]

$$\varphi(x) = 1 + x + \delta(x) + \delta'(x)$$

Hence, it is the required solution.

8.3. VOLTERRA INTEGRAL EQUATIONS OF THE FIRST KIND WITH LOGARITHMIC KERNEL:

Integral equations of the first kind with logarithmic kernel,

$$\int_0^x \varphi(t)\ln(x-t)\,dt = f(x), \quad f(0) = 0 \qquad (1)$$

can also be solved by means of the Laplace transformation.

We know that

$$x^v \risingdotseq \frac{\Gamma(v+1)}{p^{v+1}} \qquad (Re\ v > -1) \qquad (2)$$

Differentiate relation (2) with respect to v:

$$x^v lnx \risingdotseq \frac{1}{p^{v+1}}\frac{d\Gamma(v+1)}{dv} + \frac{1}{p^{v+1}}ln\frac{1}{p}\Gamma(v+1)$$

$$x^v lnx \risingdotseq \frac{\Gamma(v+1)}{p^{v+1}}\left[\frac{\frac{d\Gamma(v+1)}{dv}}{\Gamma(v+1)} + ln\frac{1}{p}\right] \qquad (3)$$

For $v = 0$, we have,

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$$

where γ is Euler's constant, and formula (3) takes the form

$$lnx \risingdotseq \frac{1}{p}(-\gamma - lnp) = -\frac{lnp+\gamma}{p} \qquad (4)$$

Let $\varphi(x) \risingdotseq \Phi(p), f(x) \risingdotseq \mathrm{F}(p)$. Taking the Laplace transform of both sides of (1) and utilizing formula (4), we get

$$-\Phi(p)\frac{lnp+}{p} = \mathrm{F}(p)$$

$$\Phi(p) = -\frac{p\mathrm{F}(p)}{lnp+} \qquad (5)$$

Let us write $\Phi(p)$ in the form

$$\Phi(p) = -\frac{p^2F(p)-f'(0)}{p(lnp+\gamma)} - \frac{f'(0)}{p(lnp+\gamma)} \qquad (6)$$

Since $f(0) = 0$, it follows that

$$p^2F(p) - f'(0) \risingdotseq f''(x) \qquad (7)$$

Let us return to formula (2) and write it in the form

$$\frac{x^v}{\Gamma(v+1)} \risingdotseq \frac{1}{p^{v+1}} \qquad (8)$$

Integrate both sides of (8) with respect to v from 0 to ∞. This yields

$$\int_0^\infty \frac{x^v}{\Gamma(v+1)}dv \risingdotseq \int_0^\infty \frac{dv}{p^{v+1}} = \frac{1}{plnp}$$

By the similarity theorem

$$\int_0^\infty \frac{x^v a^{-v}}{\Gamma(v+1)}dv \risingdotseq \frac{1}{pln(ap)} = \frac{1}{p(lnp+lna)}$$

If we put $a = e^\gamma$, then

$$\int_0^\infty \frac{x^v e^{-\gamma v}}{\Gamma(v+1)}dv \risingdotseq \frac{1}{p(lnp+\gamma)} \qquad (9)$$

Take advantage of equality (6). By virtue of (9)

$$\frac{f'(0)}{p(lnp+\gamma)} \risingdotseq f'(0)\int_0^\infty \frac{x^v e^{-\gamma v}}{\Gamma(v+1)}dv$$

Taking into account (7) and (9), the first term on the right of (6) may be regarded as a product of transforms. To find its original function, take advantage of the convolution theorem

$$\frac{p^2F(p)-f'(0)}{p(lnp+\gamma)} \risingdotseq \int_0^x f''(t)\left(\int_0^\infty \frac{(x-t)^v e^{-\gamma v}}{\Gamma(v+1)}dv\right)dt$$

Thus, the solution $\varphi(x)$ of the integral equation (1) will have the form

$$\varphi(x) = -\int_0^x f''(t)\left(\int_0^\infty \frac{(x-t)^v e^{-\gamma v}}{\Gamma(v+1)}dv\right)dt - f'(0)\int_0^\infty \frac{x^v e^{-\gamma v}}{\Gamma(v+1)}dv$$

where, γ is Euler's constant.

In particular, for $f(x) = x$ we get

$$\varphi(x) = -\int_0^\infty \frac{x^v e^{-\gamma v}}{\Gamma(v+1)}dv$$

8.4. NON-LINEAR VOLTERRA INTEGRAL EQUATIONS WITH CONVOLUTION TYPE:

The convolution theorem can also be used for solving non-linear Volterra integral equations of the type

$$\varphi(x) = f(x) + \lambda \int_0^x \varphi(t)\varphi(x-t)dt \qquad (1)$$

Let, $\varphi(x) \risingdotseq \Phi(p), f(x) \risingdotseq F(p)$

Then, by virtue of equation (1)

$$\Phi(p) = F(p) + \lambda\Phi^2(p)$$

$$\lambda\Phi^2(p) - \Phi(p) + F(p) = 0$$

Compare the quadratic equation with the general form,

$ax^2 + bx + c = 0$. Then,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

So, $\Phi(p) = \frac{1 \pm \sqrt{1-4\lambda F(p)}}{2\lambda}$

The original function of $\Phi(p)$, if it exists, will be a solution of the integral equation (1).

8.4.1. Example: Solve the integral equation,

$$\int_0^x \varphi(t)\varphi(x-t)dt = \frac{x^3}{6}$$

Solution: Consider the given integral equation,

$$\int_0^x \varphi(t)\varphi(x-t)dt = \frac{x^3}{6}$$

Compare the given equation with the general form,

$$\varphi(x) = f(x) + \lambda \int_0^x \varphi(t)\varphi(x-t)dt,$$

$$\varphi(x) \risingdotseq \Phi(p), \qquad f(x) \risingdotseq F(p)$$

Now, applying the Laplace transform on both sides of the given equation, we get,

$$L\left\{\int_0^x \varphi(t)\varphi(x-t)dt\right\} = L\left\{\frac{x^3}{6}\right\}$$

$$L\{\varphi(x) * \varphi(x)\} = L\left\{\frac{x^3}{6}\right\}$$

$$L\{\varphi^2(x)\} = L\left\{\frac{x^3}{6}\right\}$$

$$\Phi^2(p) = \frac{1}{p^4} \qquad (\because \text{Properties -5.2.9})$$

$$\Phi(p) = \sqrt{\frac{1}{p^4}}$$

$$\Phi(p) = \pm\frac{1}{p^2}$$

$$L\{\varphi(x)\} = \pm\frac{1}{p^2}$$

$$\varphi(x) = \pm L^{-1}\left\{\frac{1}{p^2}\right\}$$

$$\varphi(x) = \pm x \qquad (\because \text{ Properties -5.2.10})$$

Hence, the functions $\varphi_1(x) = x$ and $\varphi_2(x) = -x$, will be solutions of the given equation.

8.4.2. Example: Solve the integral equation,

$$2\varphi(x) - \int_0^x \varphi(t)\varphi(x-t)dt = sinx$$

Solution: Consider the given integral equation,

$$2\varphi(x) - \int_0^x \varphi(t)\varphi(x-t)dt = sinx$$

Compare the given equation with the general form,

$$\varphi(x) = f(x) + \lambda\int_0^x \varphi(t)\varphi(x-t)dt,$$

$$\varphi(x) \risingdotseq \Phi(p), \qquad f(x) \risingdotseq F(p)$$

Now, applying the Laplace transform on both sides of the given equation, we get,

$$2L\{\varphi(x)\} - L\left\{\int_0^x \varphi(t)\varphi(x-t)dt\right\} = L\{sinx\}$$

$$2L\{\varphi(x)\} - L\{\varphi(x) * \varphi(x)\} = L\{sinx\}$$

$(\because$ Properties -5.2.9)

$$2\Phi(p) - L\{\varphi^2(x)\} = \frac{1}{1+p^2}$$

$$2\Phi(p) - \Phi^2(p) = \frac{1}{1+p^2}$$

$$\Phi^2(p) - 2\Phi(p) + \frac{1}{1+p^2} = 0$$

Compare with quadratic equation with the general form,

$$ax^2 + bx + c = 0.$$

Then,

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So, $\Phi(p) = \dfrac{2\pm\sqrt{4-4\times1\times\frac{1}{1+p^2}}}{2\times1}$

$$\Phi(p) = \frac{2\pm\sqrt{4-\frac{4}{1+p^2}}}{2}$$

$$\Phi(p) = \frac{2\pm2\sqrt{1-\frac{1}{1+p^2}}}{2}$$

$$\Phi(p) = 1 \pm \sqrt{1 - \frac{1}{1+p^2}}$$

$$\Phi(p) = 1 \pm \sqrt{\frac{p^2}{1+p^2}}$$

$$\Phi(p) = 1 \pm \frac{p}{\sqrt{1+p^2}}$$

Since $\varphi(x)$ is bounded, we take the negative sign.

Thus,

$$\Phi(p) = 1 - \frac{p}{\sqrt{1+p^2}}$$

$$\Phi(p) = \frac{\sqrt{1+p^2}-p}{\sqrt{1+p^2}}$$

Now, multiply and divide by $\sqrt{1+p^2} + p$, we get,

$$\Phi(p) = \frac{\sqrt{1+p^2}-p}{\sqrt{1+p^2}} \times \frac{\sqrt{1+p^2}+p}{\sqrt{1+p^2}+p}$$

$$\Phi(p) = \frac{1+p^2-p^2}{\left(\sqrt{1+p^2}\right)\left(\sqrt{1+p^2}+p\right)}$$

$$\Phi(p) = \frac{1}{\left(\sqrt{1+p^2}\right)\left(\sqrt{1+p^2}+p\right)}$$

$$\Phi(p) = \frac{\sqrt{1+p^2}-p}{1+p^2}$$

$$\Phi(p) = \frac{\sqrt{1+p^2}}{1+p^2} - \frac{p}{1+p^2}$$

$$\Phi(p) = \frac{1}{\sqrt{1+p^2}} - \frac{p}{1+p^2}$$

$$L\{\varphi(x)\} = \frac{1}{\sqrt{1+p^2}} - \frac{p}{1+p^2}$$

$$\varphi(x) = L^{-1}\left\{\frac{1}{\sqrt{1+p^2}} - \frac{p}{1+p^2}\right\}$$

$$\varphi(x) = L^{-1}\left\{\frac{1}{\sqrt{1+p^2}}\right\} - L^{-1}\left\{\frac{p}{1+p^2}\right\}$$

$$\varphi(x) = J_0(x) - cosx$$

$$\varphi(x) = J_1(x)$$

Hence, it is the required solution.

8.4.3. Example: Solve the integral equation,

$$\varphi(x) = \frac{1}{2}\int_0^x \varphi(t)\varphi(x-t)dt - \frac{1}{2}sinhx$$

Solution: Consider the given integral equation,

$$\varphi(x) = \frac{1}{2}\int_0^x \varphi(t)\varphi(x-t)dt - \frac{1}{2}sinhx$$

Compare the given equation with the general form,

$$\varphi(x) = f(x) + \lambda \int_0^x \varphi(t)\varphi(x-t)dt,$$

$$\varphi(x) \risingdotseq \Phi(p), \qquad f(x) \risingdotseq F(p)$$

Now, applying the Laplace transform on both sides of the given equation, we get,

$$L\{\varphi(x)\} = \frac{1}{2}L\left\{\int_0^x \varphi(t)\varphi(x-t)dt\right\} - \frac{1}{2}L\{sinhx\}$$

$$L\{\varphi(x)\} = \frac{1}{2}L\{\varphi(x) * \varphi(x)\} - \frac{1}{2}\frac{1}{p^2-1}$$

($\because$ Properties -5.2.9)

$$L\{\varphi(x)\} = \frac{1}{2}L\{\varphi^2(x)\} - \frac{1}{2}\frac{1}{p^2-1}$$

$$\Phi(p) = \frac{1}{2}\Phi^2(p) - \frac{1}{2}\frac{1}{p^2-1}$$

$$2\Phi(p) = \Phi^2(p) - \frac{1}{p^2-1}$$

$$\Phi^2(p) - 2\Phi(p) - \frac{1}{p^2-1}$$

Compare with the quadratic equation with the general form,

$ax^2 + bx + c = 0$. Then,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

So, $\Phi(p) = \dfrac{2 \pm \sqrt{4-4\times 1\times \frac{-1}{p^2-1}}}{2\times 1}$

$$\Phi(p) = \frac{2 \pm \sqrt{4+\frac{4}{p^2-1}}}{2}$$

$$\Phi(p) = \frac{2 \pm 2\sqrt{1+\frac{1}{p^2-1}}}{2}$$

$$\Phi(p) = 1 \pm \sqrt{1+\frac{1}{p^2-1}}$$

$$\Phi(p) = 1 \pm \sqrt{\frac{p^2-1+1}{p^2-1}}$$

$$\Phi(p) = 1 \pm \sqrt{\frac{p^2}{p^2-1}}$$

$$\Phi(p) = 1 \pm \frac{p}{\sqrt{p^2-1}}$$

Since, $\varphi(x)$ is bounded, we take the negative sign.

Thus,

$$\Phi(p) = 1 - \frac{p}{\sqrt{p^2-1}}$$

$$L\{\varphi(x)\} = 1 - \frac{p}{\sqrt{p^2-1}}$$

$$\varphi(x) = L^{-1}\left\{1 - \frac{p}{\sqrt{p^2-1}}\right\}$$

$$\varphi(x) = L^{-1}\{1\} - L^{-1}\left\{\frac{p}{\sqrt{p^2-1}}\right\}$$

$$\varphi(x) = \delta(x) - \frac{d}{dx}L^{-1}\left\{\frac{1}{\sqrt{p^2-1}}\right\}$$

$$\varphi(x) = \delta(x) - \frac{d}{dx}\left(I_0(x)\right)$$

$$\varphi(x) = \delta(x) - I_1(x)$$

Hence, it is the required solution.

8.5 SUMMARY:

In this section, we explore the Volterra integral equations of the first kind of the convolution type. In connection with this, we have thoroughly discussed the necessary conditions for the existence of a solution to an integral equation. Apart from this, we have discussed the Volterra integral equations of the first kind with a logarithmic kernel and non-linear Volterra integral equations with convolution type. A few examples in each category have been provided to enhance the reader's understanding.

8.6 TECHNICAL TERMS:

Volterra Integral Equations of the First Kind of the Convolution Type:

An integral equation of the first kind,

$$\int_0^x K(x-t)\varphi(t)dt = f(x) \qquad (1)$$

whose kernel $K(x,t)$ is dependent solely on the difference $(x-t)$ of arguments will be called an integral equation of the first kind of the convolution type.

Necessary condition for the existence of a solution of an integral equation: A necessary condition for the existence of a continuous solution of an integral equation of the form,

$$\int_0^x \frac{(x-t)^{n-1}}{(n-1)!}\varphi(t)dt = f(x)$$

Non-linear Volterra Integral Equations with Convolution Type: The non-linear Volterra integral equation with convolution type is of the form,

$$\varphi(x) = f(x) + \lambda\int_0^x \varphi(t)\varphi(x-t)dt \qquad (1)$$

8.7 SELF-ASSESSMENT QUESTIONS:

Exercise (8.1): Solve the integral equations:

(1) $\int_0^x e^{x-t}\varphi(t)dt = sinhx$

(2) $\int_0^x e^{x-t}\varphi(t)dt = x^2$

(3) $\int_0^x \cos(x-t)\varphi(t)dt = xsinx$

(4) $\int_0^x \sinh(x-t)\varphi(t)dt = x^3e^{-x}$

(5) $\int_0^x J_o(x-t)\varphi(t)dt = sinx$

(6) $\int_0^x \cosh(x-t)\varphi(t)dt = x$

(7) $\int_0^x (x^2-t^2)\varphi(t)dt = \frac{x^3}{3}$

(8) $\int_0^x (x^2-4xt+3t^2)\varphi(t)dt = \frac{x^4}{12}$

(9) $\frac{1}{2}\int_0^x (x^2-4xt+3t^2)\varphi(t)dt = x^2 J_4(2\sqrt{x})$

(10) $\int_0^x (x-2t)\varphi(t)dt = \frac{-x^3}{6}$

(11) $\int_0^x \sin(x-\mathrm{t})\varphi(t)dt = 1 - cosx$

Solutions to Exercise(8.1):

(1) $\varphi(x) = e^{-x}$

(2) $\varphi(x) = 2x - x^2$

(3) $\varphi(x) = 2\sin x$

(4) $\varphi(x) = 3!\,(xe^{-x} - x^2e^{-x})$

(5) $\varphi(x) = J_o(x)$

(6) $\varphi(x) = 1 - \frac{x^2}{2}$

(7) $\varphi(x) = \frac{1}{2}$

(8) $\varphi(x) = \mathrm{C} - x$

(9) $\varphi(x) = C + J_0(2\sqrt{x})$

(10) $\varphi(x) = \mathrm{C} + x$

(11) $\varphi(x) = 1$

8.8 SUGGESTED READINGS:

1. M. D. Raisinghania, Integral Equations and Boundary Value Problems, S. Chand and Company Pvt. Ltd., 2007.
2. Shanti Swarup, Integral Equations, Krishna Prakashan Pvt Ltd, Meerut, 2003.
3. M. Krasnov, A. Kiselev, G. Makarenko, Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971.
4. M. Rahman, Integral Equations and Their Applications, WIT Press, Southampton, Boston, 2007.

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LESSON- 9

FREDHOLM EQUATIONS OF THE SECOND KIND

OBJECTIVE:

- To understand the fundamental concepts of Fredholm integral equations and their classifications.
- To explore important theorems, such as Fredholm's theorem and its implications.
- To learn various solution methods, including the use of resolvent kernels and numerical techniques.

STRUCTURE:

9.1 Introduction

9.2 Fundamentals

9.3 The Method of Fredholm Determinants Resolvent Kernels

9.4 Summary

9.5 Technical Terms

9.6 Self-Assessment Questions

9.7 Suggested Readings

9.1 INTRODUCTION:

Fredholm integral equations, introduced in 1903 by Swedish mathematician Erik Ivar Fredholm, are fundamental in mathematical physics, engineering, and applied sciences. These equations, which express unknown functions through integrals involving a kernel function, laid the groundwork for modern functional analysis and operator theory. They are categorized into two types: the first kind, where the unknown appears only under the integral, and the second kind, where it appears both inside and outside the integral. Widely applied in quantum mechanics, signal processing, and boundary value problems, Fredholm equations have driven advances in both analytical and numerical methods for solving complex equations, making them essential in fields like heat conduction, fluid dynamics, and computational mechanics

The study of Fredholm integral equations has led to significant developments in numerical and analytical methods for solving complex integral and differential equations. Today, these equations are indispensable tools in various scientific and engineering applications, including heat conduction, potential theory, and fluid dynamics, computational mechanics.

9.2 FUNDAMENTALS:

Now we recollect some important definitions and examples which are essential in the study of this Lesson.

9.2.1 Definition: A *Fredholm integral equation* is an integral equation in which the unknown function $\varphi(x)$ appears inside an integral over a finite domain $[a, b]$. It is classified into two types:

Fredholm integral equation of the first kind:

$$\int_a^b K(x,t)\varphi(t)dt = f(x) \tag{9.1}$$

Fredholm integral equation of the second kind:

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x), \tag{9.2}$$

where $\varphi(x)$ is an unknown function, $K(x,t)$ and $f(x)$ are known functions, x and t are real variables varying in the interval (a, b), and λ is a numerical factor. The function $K(x,t)$ is called the kernel of the integral equations (9.1) and (9.2).

If $f(x) \not\equiv 0$, equation (9.2) is *nonhomogeneous*. However, if $f(x) \equiv 0$, then (9.2) takes the form

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = 0, \tag{9.3}$$

which is called *homogeneous* Fredholm integral equation of the second kind. The limits of integration, a and b in equations (9.1), (9.2) and (9.3) can be either finite or infinite.

9.2.2 Show that the function $\varphi(x) = sin\frac{\pi x}{2}$ is a solution of the Fredholm-type integral equation

$$\varphi(x) - \frac{\pi^2}{4}\int_0^1 K(x,t)\varphi(t)dt = \frac{x}{2}$$

where the kernel is of the form

$$K(x,t) = \begin{cases} \dfrac{x(2-t)}{2}, & 0 \le x \le t, \\ \dfrac{t(2-x)}{2}, & t \le x \le 1. \end{cases}$$

Solution. Write the left-hand side of the equation as

$L.H.S. = \varphi(x)$

$$-\frac{\pi^2}{4}\int_0^1 K(x,t)\,\varphi(t)\,dt = \varphi(x) - \frac{\pi^2}{4}\left\{\int_0^x K(x,t)\,\varphi(t)\,dt + \int_x^1 K(x,t)\,\varphi(t)\,dt\right\}$$

$$= \varphi(x) - \frac{\pi^2}{4}\left\{\int_0^x \frac{t(2-x)}{2}\,\varphi(t)\,dt + \int_x^1 \frac{x(2-t)}{2}\,\varphi(t)\,dt\right\}$$

$$= \varphi(x) - \frac{\pi^2}{4}\left\{\frac{2-x}{2}\int_0^x t\,\varphi(t)\,dt + \frac{x}{2}\int_x^1 (2-t)\,\varphi(t)\,dt\right\}$$

Substituting the function $sin\frac{\pi x}{2}$ in place of $\varphi(x)$ in to this expression, we get

$$sin\frac{\pi x}{2} - \frac{\pi^2}{4}\left\{(2-x)\int_0^x t\,\frac{sin\frac{\pi t}{2}}{2}\,dt + x\int_x^1 (2-t)\,\frac{sin\frac{\pi t}{2}}{2}dt\right\} =$$

$$= sin\frac{\pi x}{2} - \frac{\pi^2}{4}\left\{(2-x)\left(-\frac{t}{\pi}cos\frac{\pi t}{2} + \frac{2}{\pi^2}sin\frac{\pi t}{2}\right)\Bigg|_{t=0}^{t=x} + x\left[-\frac{2-t}{\pi}cos\frac{\pi t}{2} - \frac{2}{\pi^2}sin\frac{\pi t}{2}\right]\Bigg|_{t=x}^{t=1}\right\}$$

$$= \frac{x}{2} = R.H.S.$$

Thus, $\varphi(x) = sin\frac{\pi x}{2}$ is a solution of the given integral equation.

9.2.3 Example: Show that the function $\varphi(x) = 1$ is a solution of the Fredholm-type integral equation

$$\varphi(x) + \int_0^1 x(e^{xt}-1)\varphi(t)dt = e^x - x$$

Solution. Write the left-hand side of the equation as

$$L.H.S. = \varphi(x) + \int_0^1 x(e^{xt}-1)\varphi(t)dt = 1 + \int_0^1 x(e^{xt}-1)(1)dt = 1 + x\int_0^1 (e^{xt}-1)dt$$

$$= 1 + x\left(\int_0^1 (e^{xt}-1)dt\right) = 1 + x\left(\frac{e^x-1-x}{x}\right) = e^x - x = R.H.S.$$

Therefore $\varphi(x) = 1$ is a solution of the given integral equation.

9.3 THE METHOD OF FREDHOLM DETERMINANTS AND RESOLVENT KERNELS:

Here we can observe the definitions of Fredholm resolvent kernel, Fredholm minor, Fredholm determinant and finding the solution of the Fredholm equation of second kind as follows:

9.3.1 Definition: The solution of the Fredholm equation of the second kind

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x), \tag{9.4}$$

is given by the formula

$$\varphi(x) = f(x) + \lambda \int_a^b R(x,t;\ \lambda) f(t)dt, \tag{9.5}$$

where the function $R(x,t;\lambda)$ is called the $Fredholm\ resolvent\ kernel$ of equation (9.4) and is defined by the equation

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}, \tag{9.6}$$

provided that $D(\lambda) \neq 0$.

Note: In the equation (9.6), $D(x,t;\lambda)$ and $D(\lambda)$ are expressed as power series in λ as given below.

$$D(x,t;\lambda) = K(x,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(x,t)\lambda^n, \tag{9.7}$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C_n(x,t)\lambda^n, \tag{9.8}$$

whose coefficients are given by the formulas

$$B_n(x,t) = \int_a^b \cdots \int_a^b \begin{vmatrix} K(x,t) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ K(t_2,t) & K(t_2,t_1) & \cdots & K(t_2,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

and

$$B_0(x,t) = K(x,t)$$

$$C_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \cdots & K(t_1,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \cdots & K(t_2,t_n) \\ K(t_3,t_1) & K(t_3,t_2) & \cdots & K(t_3,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t_1) & K(t_n,t_2) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n$$

9.3.2 Definition: The function $D(x,t;\lambda)$ as defined in the equation (9.7) is called $Fredholm\ minor$, and $D(\lambda)$ is called $Fredholm\ determinant$.

Note:

- If the kernel $K(x,t)$ is bounded or the integral
$$\int_a^b \int_a^b K^2(x,t)\,dx\,dt$$
has a finite value, then the functions $D(\lambda)$ and $D(x,t{:}\lambda)$ are converge for all values of λ and, hence, these functions are entire analytic functions of λ.
- The resolvent kernel
$$R(x,t;\lambda) = \frac{D(x,t;\ \lambda)}{D(\lambda)}$$
is analytic function of λ, except for those values of λ where $D(\lambda) = 0$; These values are called poles of $R(x,t;\ \lambda)$.

9.3.3 Example: Using Fredholm determinants, find the resolvent kernel of the kernel $K(x,t) = xe^t$; $a = 0, b = 1$.

Solution. We have $B_0(x,t) = xe^t$ Further,

$$B_1(x,t) = \int_0^1 \begin{vmatrix} xe^t & xe^{t_1} \\ t_1e^t & t_1e^{t_1} \end{vmatrix} dt_1 = 0,$$

$$B_2(x,t) = \int_0^1 \int_0^1 \begin{vmatrix} xe^t & xe^{t_1} & xe^{t_2} \\ t_1e^t & t_1e^{t_1} & t_1e^{t_2} \\ t_2e^t & t_2e^{t_1} & t_2e^{t_2} \end{vmatrix} dt_1dt_2 = 0$$

since the determinants under the integral sign are zero. It is obvious that all subsequent $B_n(x,t) = 0$. Now we find the coefficients C_n:

$$C_1 = \int_0^1 K(t_1,t_1)dt_1 = \int_0^1 t_1e^{t_1}dt_1 = 1,$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} t_1e^{t_1} & t_1e^{t_2} \\ t_2e^{t_1} & t_2e^{t_2} \end{vmatrix} dt_1dt_2 = 0$$

Obviously, all subsequent C_n are also equal to zero.

In our case, by the formulas of $D(x,t;\lambda)$ and $D(\lambda)$, we have

$$D(x,t;\lambda) = K(x,t) = xe^t; \quad D(\lambda) = 1-\lambda$$

Thus,

$$R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)} = \frac{xe^t}{1-\lambda}$$

Let us apply the result obtained to solving the integral equation

$$\varphi(x) - \lambda \int_0^1 xe^t \varphi(t)dt = f(x) \ \ (\lambda \neq 1)$$

By using the formula (9.5) we get,

$$\varphi(x) = f(x) + \lambda \int_0^1 \frac{xe^t}{1-\lambda} f(t)dt$$

In particular, for $f(x) = e^{-x}$ we get

$$\varphi(x) = e^{-x} + \frac{\lambda}{1-\lambda} x.$$

9.3.4 Example: Using Fredholm determinants, find the resolvent kernel of the kernel $K(x,t) = 2x - t;\ 0 \le x \le 1, 0 \le t \le 1.$

Solution. We have $B_0(x,t) = 2x - t.$

Further,

$$B_1(x,t) = \int_0^1 \begin{vmatrix} 2x-t & 2x-t_1 \\ 2t_1 - t & t_1 \end{vmatrix} dt_1 = -x + 2xt + \frac{2}{3} - t,$$

$$B_2(x,t) = \int_0^1 \int_0^1 \begin{vmatrix} 2x-t & 2x-t_1 & 2x-t_2 \\ 2t_1-t & t_1 & 2t_1-t_2 \\ 2t_2-t & 2t_2-t_1 & t_2 \end{vmatrix} dt_1 dt_2 = 0$$

In a similar way, all subsequent $B_n(x,t) = 0$. Now we find the coefficients C_n:

$$C_1 = \int_0^1 K(t_1,t_1)dt_1 = \int_0^1 2t_1 - t_1 dt_1 = \frac{1}{2},$$

$$C_2 = \int_0^1 \int_0^1 \begin{vmatrix} t_1 & 2t_1 - t_2 \\ 2t_2 - t_1 & t_2 \end{vmatrix} dt_1 dt_2 = \frac{1}{3}$$

$$C_3 = \int_0^1 \int_0^1 \int_0^1 \begin{vmatrix} t_1 & 2t_1-t_2 & 2t_1-t_3 \\ 2t_2-t_1 & t_2 & 2t_2-t_3 \\ 2t_3-t_1 & 2t_3-t_2 & t_3 \end{vmatrix} dt_1 dt_2 dt_3 = 0$$

Obviously, all subsequent C_n are also equal to zero.

In our case, by the formulas of $D(x,t;\lambda)$ and $D(\lambda)$, we have

$$D(x,t;\lambda) = (2x-t) + \lambda(x - 2xt - \frac{2}{3} + t);$$

$$D(\lambda) = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}.$$

Thus,

$$R(x,t,\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)} = \frac{2x - t + \lambda(x - 2xt - \frac{2}{3} + t)}{1 - \frac{\lambda}{2} + \frac{\lambda^2}{6}}.$$

9.3.5 Note: In very rare cases is it possible to compute the coefficients $B_n(x,t)$ and C_n of the series $D(x,t;\lambda)$ and $D(\lambda)$, However, these formulas it is possible to obtain the following recursion relations:

$$B_n(x,t) = C_n K(x,t) - n\int_a^b K(x,s)B_{n-1}(s,t)ds, \tag{9.9}$$

$$C_n = \int_a^b B_{n-1}(s,s)ds \tag{9.10}$$

Where, the coefficient $C_0 = 1$ and $B_0(x,t) = K(x,t)$, we can use formulas (9.9) and (9.10) to successively compute the next terms $C_{1,}\ B_1(x,t), C_{2,}\ B_2(x,t), C_3$ and so on.

9.3.6 Example: Using formulas (9.9) and (9.10), find the resolvent kernel of the kernel $K(x,t) = x - 2t$, where $0 \leq x \leq 1, 0 \leq t \leq 1$.

Solution. We have $C_0 = 1$ and $B_0(x,t) = x - 2t$. Using formula (9.10), we find

$$C_1 = \int_0^1 (-s)ds = -\frac{1}{2}$$

By formula (9.9) we get

$$B_1(x,t) = -\frac{x-2t}{2} - \int_0^1 (x-2s)\,(s-2t)ds = -x - t + 2xt + \frac{2}{3}$$

We further obtain

$$C_2 = \int_0^1 \left(-2s + 2s^2 + \frac{2}{3}\right) ds = \frac{1}{3}$$

$$B_2(x,t) = \frac{x-2t}{3} - 2\int_0^1 (x-2s)\left(-s - t + 2st + \frac{2}{3}\right) ds = 0$$

$$C_3 = C_4 = \cdots = 0, B_3(x,t) = B_4(x,t) = \cdots = 0$$

Hence,

$$D(\lambda) = 1 + \frac{\lambda}{2} + \frac{\lambda^2}{6};\ \ D(x,t;\lambda) = x - 2t + \left(x + t - 2xt - \frac{2}{3}\right)\lambda$$

The resolvent kernel of the given kernel is

$$R(x,t;\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)} = \frac{x - 2t + \left(x + t - 2xt - \frac{2}{3}\right)\lambda}{1 + \frac{\lambda}{2} + \frac{\lambda^2}{6}}.$$

9.3.7 Example: Using the recursion relations (9.9) and (9.10), find the resolvent kernels of the kernel $K(x,t) = x + t + 1;\quad -1 \leq x \leq 1,\ \ -1 \leq t \leq 1.$

Solution. We have $C_0 = 1$ and $B_0(x,t) = x + t + 1$. Using formula (9.10), we find

$$C_1 = \int_{-1}^{1} (2s+1)ds = 2.$$

By formula (9.9) we get

$$B_1(x,t) = 2(x+t+1) - \int_{-1}^{1} (x+s+1)\,(s+t+1)ds = -2xt - \frac{2}{3}$$

We further obtain

$$C_2 = \int_{-1}^{1} \left(-2s^2 - \frac{2}{3}\right) ds = -\frac{8}{3}$$

$$B_2(x,t) = -\frac{8}{3}(x+t+1) - \int_{-1}^{1} (x+s+1)\left(-2st - \frac{2}{3}\right) ds = 0$$

$$C_3 = C_4 = \cdots = 0, B_3(x,t) = B_4(x,t) = \cdots = 0$$

Hence,

$$D(\lambda) = 1 - 2\lambda - \frac{4\lambda^2}{3}$$

$$D(x,t;\lambda) = x+t+1+2\left(xt+\frac{1}{3}\right)\lambda$$

The resolvent kernel of the given kernel is

$$R(x,t;\lambda) = \frac{D(x,t,\lambda)}{D(\lambda)} = \frac{x+t+1+2\left(xt+\frac{1}{3}\right)\lambda}{1-2\lambda-\frac{4\lambda^2}{3}}.$$

9.4 SUMMARY:

This unit provides the fundamental idea of the Fredholm integral equations of first and second kind. The method of Fredholm determinants is used to find the resolvent kernels that helps us in obtaining the solutions of various integral equations.

9.5 TECHNICAL TERMS:

- **Fredholm Integral Equation of the Second Kind:** An equation of the form

$$\int_a^b K(x,t)\varphi(t)dt = f(x),$$

Where the unknown function appears both inside and outside the integral.

- **Kernel:** The kernel $K(x,t)$ is a given function in a Fredholm integral equation and is referred to as the kernel of the equation.
- **Homogeneous Integral Equation:** A Fredholm equation where $f(x) = 0$, resulting

$$\int_a^b K(x,t)\varphi(t)dt = 0,$$

is called homogeneous integral equation.

- **Non-Homogeneous Integral Equation:** A Fredholm equation where $f(x) \neq 0$, so the right-hand side remains a known non-zero function**.**
- **Fredholm Determinant:** A special function $D(\lambda)$, constructed using the kernel, used to determine whether the equation has a unique solution.
- **Fredholm Minor:** The function $D(x,t{:}\lambda)$ that appears in the definition of the resolvent kernel
- **Resolvent Kernel:** The function $R(x,t;\lambda) = \frac{D(x,t:\lambda)}{D(\lambda)}$, which helps solve the integral equation explicitly.

9.6 SELF-ASSESSMENT QUESTIONS:

Exercise (9a): check whether the given functions are the solutions of the indicated integral equations:

1. $\varphi(x) = e^x\left(2x - \frac{2}{3}\right)$,

$$\varphi(x) + 2\int_0^1 e^{x-t}\,\varphi(t)\,dt = 2xe^x.$$

2. $\varphi(x) = 1 - \frac{2\sin x}{1-\frac{\pi}{2}}$,

$$\varphi(x) - \int_0^\pi \cos(x+t)\,\varphi(t)\,dt = 1.$$

3. $\varphi(x) = \sqrt{x}$,

$$\varphi(x) - \int_0^1 K(x,t)\varphi(t)dt = \sqrt{x} + \frac{x}{15}\left(4x^{3/2} - 7\right).$$

$$K(x,t) = \begin{cases} \dfrac{x(2-t)}{2}, & 0 \le x \le t, \\ \dfrac{t(2-x)}{2}, & t \le x \le 1. \end{cases}$$

4. $\varphi(x) = e^x$,

$$\varphi(x) + \lambda \int_0^1 \sin xt\, \varphi(t)dt = 1.$$

5. $\varphi(x) = \cos x$,

$$\varphi(x) - \int_0^\pi (x^2 + t)\cos t\ \varphi(t)dt = \sin x\,.$$

6. $\varphi(x) = xe^{-x}$,

$$\varphi(x) - 4\int_0^\infty e^{-(x+t)}\varphi(t)dt = (x-1)e^{-x}.$$

7. $\varphi(x) = \cos 2x$,

$$\varphi(x) - 3\int_0^\pi K(x,t)\varphi(t)dt = \cos x\,.$$

$$K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x \le t, \\ \sin t \cos x\,, & t \le x \le \pi. \end{cases}$$

8. $\varphi(x) = \frac{4C}{\pi}\sin x$, where c is an arbitrary constant,

$$\varphi(x) - \frac{4}{\pi}\int_0^\infty \sin x \frac{\sin^2 t}{t}\varphi(t)dt = 0.$$

Exercise (9b):

Using the Fredholm determinants, find the resolvent kernels of the following kernels:

1. $K(x,t) = x^2 t - xt^2$; $0 \le x,t \le 1$.
2. $K(x,t) = sin\ x\ cos\ t$; $0 \le x,t \le 2\pi$.
3. $K(x,t) = sin\ x - sin\ t$; $0 \le x,t \le 2\pi$.

Using the recursion relations (9.9) and (9.10), find the resolvent kernels of the following kernels:

4. $K(x,t) = 1 + 3xt$; $0 \le x,t \le 1$.
5. $K(x,t) = 4xt - x^2$; $0 \le x,t \le 1$.
6. $K(x,t) = e^{x-t}$; $0 \le x,t \le 1$.
7. $K(x,t) = \sin(x+t)$ $0 \le x,t \le 2\pi$.
8. $K(x,t) = x - sin\,h\,t\,; -1 \le x,t \le 1$.

Exercise (9c):

Using the resolvent kernel, solve the following integral equations**:**

1.
$$\varphi(x) - \lambda \int_0^{2\pi} sin\ (x+t)\varphi(t)dt = 1$$

2.
$$\varphi(x) - \lambda \int_0^{1} (2x-t)\varphi(t)dt = \frac{x}{6}$$

3.
$$\varphi(x) - \int_0^{2\pi} sin\ x\ cos\ \ t\ \varphi(t)dt = cos\ \ 2x$$

4.
$$\varphi(x) + \int_0^{1} e^{x-t}\varphi(t)dt = e^{x}.$$

5.
$$\varphi(x) - \lambda \int_0^{1} (4xt - x^2)\ \varphi(t)dt = x.$$

9.7 SELF-ASSESSMENT ANSWERS:

Exercise (9b)

1. $R(x,t;\lambda) = \frac{x^2t - xt^2 + x\ \left(\frac{x+t}{4} - \frac{xt}{3} - \frac{1}{5}\right)\lambda}{1 + \frac{\lambda^2}{240}}.$
2. $R(x,t;\lambda) = sinx\ cost.$
3. $R(x,t;\lambda) = \frac{\sin x - \sin t - \ \ (1 + 2\sin x \sin\ \ )\lambda.}{1 + 2\pi^2\lambda^2}$
4. $R(x,t;\lambda) = \frac{1 + 3xt + \left(3\frac{x+t}{2} - 3xt - 1\right)\lambda}{1 - 2\lambda + \frac{1}{4}\lambda^2}.$
5. $R(x,t;\lambda) \ = \frac{4xt - x^2 - \left(2x^2t - \frac{4}{3}x^2 + x - \frac{4}{3}xt\right)\lambda}{1 - \lambda + \frac{\lambda^2}{18}}.$
6. $R(x,t;\lambda) = \frac{e^{x-t}}{1-\lambda}.$
7. $R(x,t;\lambda) = \frac{\sin(x+t) + \pi\lambda \cos\ (x-t)}{1 - \pi^2\lambda^2}.$
8. $R(x,t;\lambda) = \frac{x - \sinh t - 2\left(e^{-1} + x\sinh t\right)\lambda}{1 + 4e^{-1}\lambda^2}.$

Exercise (9c)

1. $\phi(x) = 1.$
2. $\phi(x) = \frac{1}{6}\left[x + \frac{(6x-2)\lambda - \lambda^2 x}{\lambda^2 - 3\lambda + 6}\right].$
3. $\phi(x) = \cos 2x.$
4. $\phi(x) = \frac{1}{2}e^x.$
5. $\phi(x) = \frac{3x(2\lambda - 3\lambda x + 6)}{\lambda^2 - 18\lambda + 18}.$

9.8 SUGGESTED READINGS:

1. K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering*, Cambridge University Press, 2006 (Third Edition). ISBN-978-0521679718.
2. F. G. Tricomi, *Integral Equations*, Dover Publications, 1985. ISBN-978-0486648286.
3. Rainer Kress, *Linear Integral Equations*, Springer, 2014 (Third Edition). ISBN-978-1447171474.
4. I. G. Petrovsky, *Lectures on Partial Differential Equations*, Dover Publications, 2012. ISBN-978-0486659640.
5. M. A. Krasnoselskii, *Integral Equations of the First Kind: Invariant Imbedding Method and Applications*, CRC Press, 1994. ISBN-978-2884490651.

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LESSON- 10
ITERATED KERNELS

OBJECTIVE:

- To understand the concept and significance of iterated kernels in the context of Fredholm integral equations.
- To learn the method of constructing iterated kernels and their use in building the resolvent kernel
- To explore the convergence criteria for Neumann series and conditions under which a solution exists.
- To study examples that illustrate the construction and application of iterated kernels.

STRUCTURE:

10.1 Introduction

10.2 Definition and Formation of Iterated Kernels

10.3 Construction of the Resolvent Kernel using Iterated Kernels and Convergence of Neumann Series

10.4 Construction of the Resolvent Kernel to the Orthogonal kernels

10.5 Summary

10.6 Technical Terms

10.7 Self-Assessment Questions

10.8 Suggested Readings

10.1 INTRODUCTION:

In integral equations, iterated kernels play a key role in the construction of the resolvent kernel, which is used to obtain solutions to Fredholm integral equations of the second kind. By defining successive approximations of the solution, we form a series where each term involves an iterated kernel derived from the original kernel. These concepts provide powerful analytical tools for examining the behavior and solvability of integral equations under various conditions.

10.2 DEFINITION AND FORMATION OF ITERATED KERNELS:

We define degenerate kernels and formulate the integral equation using degenerate kernels.

10.2.1 Formation of Iterated Kernels: Consider the Fredholm integral equation of second kind

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x). \qquad (10.1)$$

As in the case of the Volterra equations, the integral equation (10.1) may be solved by the method of successive approximations. In order to solve the equation, we represent $\varphi(x)$ as follows:

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} \psi_n(x)\lambda^n, \tag{10.2}$$

Where the function $\psi_n(x)$ are determined from the formulas

$$\psi_1(x) = \int_a^b K(x,t)f(t)dt,$$

$$\psi_2(x) = \int_a^b K(x,t)\,\psi_1(t)dt = \int_a^b K_2(x,t)f(t)dt,$$

$$\psi_3(x) = \int_a^b K(x,t)\,\psi_2(t)dt = \int_a^b K_3(x,t)f(t)dt,$$

and so on. Here

$$K_2(x,t) = \int_a^b K(x,z)K_1(z,t)dz,$$

$$K_3(x,t) = \int_a^b K(x,z)K_2(z,t)dz,$$

and generally,

$$K_n(x,t) = \int_a^b K(x,z)K_{n-1}(z,t)dz, \tag{10.3}$$

$n = 2, 3, \ldots$, and $K_1(x,t) \equiv K(x,t)$.

10.2.2 Definition: The functions $K_n(x,t)$ determined from

$K_n(x,t) = \int_a^b K(x,z)K_{n-1}(z,t)dz$ are called *iterated Kernels* and the following relation holds for these functions

$$K_n(x,t) = \int_a^b K_m(x,s)K_{n-m}(s,t)ds, \tag{10.4}$$

where *m* is any natural number less than *n*.

10.3 CONSTRUCTION OF THE RESOLVENT KERNEL USING ITERATED KERNELS AND CONVERGENCE OF NEUMANN SERIES

We shall construct the resolvent kernel of the integral equation (10.1) is determined in terms of iterated kernels by using the formula

$$R(x,t;\ \lambda) = \sum_{n=1}^{\infty} K_n(x,t)\lambda^{n-1}. \tag{10.5}$$

10.3.1 Definition: The series on the right side of the resolvent kernel $R(x,t;\ \lambda) = \sum_{n=1}^{\infty} K_n(x,t)(x)\lambda^{n-1}$ is the *Neumann series of the kernel* $K(x,t)$. It converges for

$$|\lambda| < \frac{1}{B}, \tag{10.6}$$

where $B = \sqrt{\int_a^b \int_a^b K^2(x,t)dxdt}$.

10.3.2 Note:

1. The solution of the Fredholm equation of the second kind (10.1) is expressed by the formula

$$\varphi(x) = f(x) + \lambda \int_a^b R(x,t;\ \lambda)f(t)dt. \qquad (10.$$

The boundary (10.6) is essential for convergence of the series (10.5). However, a solution of equation (10.1) can exist for values of $|\lambda| > \frac{1}{B}$ as well. Consider an example as

$$\varphi(x) - \lambda \int_0^1 \varphi(t)dt = 1 \qquad (10.$$

Here $K(x,t) \equiv 1$, and hence

$$B^2 = \int_0^1 \int_0^1 K^2(x,t)dx\ dt = \int_0^1 \int_0^1 dx\ dt = 1$$

Thus, the condition (10.6) gives that the series (10.5) converges for $|\lambda| < 1$. Solving (10.8) as an equation with a degenerate kernel, we get $(1-\lambda)C = 1$,

where $C = \int_0^1 \varphi(t)dt$. For $\lambda = 1$ the integral equation (10.8) does not have any solution. However, equation (10.8) is solvable for $|\lambda| > 1$. Indeed, if $\lambda \neq 1$, then the function $\varphi(x) = \frac{1}{1-\lambda}$ is a solution of the given equation.

2. From the above discussion we can observe that in a circle of radius greater than unity, successive approximation cannot converge for (10.8)
3. For some Fredholm equations the Neumann series (10.5) converges for the resolvent kernel for any values of λ

10.3.3 Definition: Let $K(x,t)$ and $L(x,t)$ are two kernels are said to be *orthogonal,* if the following two conditions are satisfied for any admissible values of x and t:

$$\int_a^b K(x,z)L(z,t)dz = 0, \quad \int_a^b L(x,z)K(z,t)dz = 0. \tag{10.9}$$

Example: The kernels $K(x,t) = xt$ and $L(x,t) = x^2t^2$ are orthogonal on $[-1,1]$.

Certainly,

$$\int_{-1}^{1} (xz)(z^2t^2)dz = xt^2 \int_{-1}^{1} z^3 dz = 0,$$

$$\int_{-1}^{1} (x^2z^2)(zt)dz = x^2t \int_{-1}^{1} z^3 dz = 0.$$

10.3.4 Note: There exist kernels which are orthogonal to themselves. For such kernels, $K_2(x,t) \equiv 0$, where $K_2(x,t)$ is the second iterated kernel. It is obvious that in this case all subsequent iterated kernels are also equal to zero and the resolvent kernel coincides with the kernel $K(x,t)$. We can observe from following example

10.3.5 Example: $K(x,t) = \sin(x-2t)$; is orthogonal itself where $0 \le x, t \le 2\pi$.

We have

$$\int_0^{2\pi} \sin(x-2z)\sin(z-2t)\,dz$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(x+2t-3z) - \cos(x-2t-z)]\,dz$$

$$= \frac{1}{2}\left[-\frac{1}{3}\sin(x+2t-3z) + \sin(x-2t-z)\right]\Bigg|_{z=0}^{z=2\pi} = 0.$$

Thus, in this case the resolvent kernel of the kernel is equal to the kernel itself:

$$R\,(x,t;\,\lambda) \equiv sin\,(x-2t)$$

so that the Neumann series (10.5) consists of one term and, obviously, converges for any λ.

10.3.6 Note: The iterated kernels $K_n(x,t)$ can be expressed directly in terms of the given kernel $K\ (x,t)$ by the formula

$$K_n(x,t)=\int_a^b\int_a^b\dots\int_a^b K(x,s_1)K_1(s_1,s_2)\dots K(s_{n-1},t)ds_1\dots ds_{n-1} \qquad (10.1$$

- All iterated kernels $K_n(x,t)$, beginning with $K_2(x,t)$ will be continuous functions in the square $a\le x\le\ b,\ a\le t\le\ b$ if the initial kernel $K(x,t)$ is quadratically summable in this square.
- If the given kernel $K(x,t)$ is symmetric, then all iterated kernels $K_n(x,t)$ are also symmetric.

10.4 CONSTRUCTION OF THE RESOLVENT KERNEL TO THE ORTHOGONAL KERNELS:

The following are some examples in finding iterated kernels.

10.4.1 Find the iterated kernels for the kernel $K(x,t)=x-t$ where $a=0,\ b=1$.

Solution. Using formulas (10.3), we find in succession as follows:

$K_1(x,t)=x-t,$

$$K_2(x,t)=\int_0^1(x-s)\,(s-t)ds=\frac{x+t}{2}-xt-\frac{1}{3},$$

$$K_3(x,t)=\int_0^1(x-s)\left(\frac{s+t}{2}-st-\frac{1}{3}\right)ds=-\frac{x-t}{12},$$

$$K_4(x,t)=-\frac{1}{12}\int_0^1(x-s)\,(s-t)ds=-\frac{1}{12}K_2(x,t)=-\frac{1}{12}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right),$$

$$K_5(x,t)=-\frac{1}{12}\int_0^1(x-s)\left(\frac{s+t}{2}-st-\frac{1}{3}\right)ds=-\frac{1}{12}K_3(x,t)=\frac{x-t}{12^2},$$

$$K_6(x,t)=\frac{1}{12^2}\int_0^1(x-s)\,(s-t)ds=\frac{K_2(x,t)}{12^2}=\frac{1}{12^2}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right).$$

From this it follows that iterated kernels are of the form:

(1) for $n=2k-1$

$$K_{2k-1}(x,t)=\frac{(-1)^{k}}{12^{k-1}}\ (x-t)$$

(2) for $n=2k$

$$K_{2k}(x,t)=\frac{(-1)^{k-1}}{12^{k-1}}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right)$$

where $k\ =\ 1,2,3,\dots$

10.4.2 Find the iterated kernels $K_1(x,t)$ and $K_2(x,t)$ if

$K(x,t) = e^{\min(x,\ t)},\ a = 0,\ b = 1.$

Solution. By definition we have

$$min(x,t) = \begin{cases} x, & if\ 0 \le x \le t, \\ t, & if\ t \le x \le 1 \end{cases}$$

and for this reason, the given kernel may be written as

$$K(x,t) = \begin{cases} e^x, & if\ 0 \le x \le t, \\ e^t, & if\ t \le x \le 1 \end{cases}$$

This kernel, as may easily be verified, is symmetric, i.e.,

$$K(x,t) = K(t,x)$$

We have $K_1(x,t) = K(x,t)$. We find the second iterated kernel

$$K_2(x,t) = \int_0^1 K(x,s)K_1(s,t)ds = \int_0^1 K(x,s)K(s,t)ds$$

Here

$$K(x,s) = \begin{cases} e^x, & if\ 0 \le x \le s, \\ e^s, & if\ s \le x \le 1 \end{cases}$$

$$K(s,t) = \begin{cases} e^s, & if\ 0 \le s \le t, \\ e^t, & if\ t \le s \le 1 \end{cases}$$

Since the given kernel $K(x,t)$ is symmetric, it is sufficient to find $K_2(x,t)$ only for $x > t$.

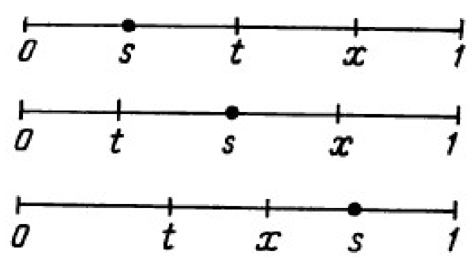

Fig. 2

From Fig 2, we have to find $K_2(x,t)$ as follows:

$$\int_t^x K(x,s)K(s,t)ds = \int_t^x e^s e^t ds = e^{x+t} - e^{2t}$$

$$K_2(x,t) = \int_0^t K(x,s)K(s,t)ds + \int_t^x K(x,s)K(s,t)ds + \int_x^1 K(x,s)K(s,t)ds.$$

In the interval $(0,t)$ we have $s<t<x$, and therefore

$$\int_0^t K(x,s)K(s,t)ds=\int_0^t e^s e^s ds=\int_0^t e^{2s}ds=\frac{e^{2t}-1}{2}$$

In the interval (t,x) we have $t<s<x$, and therefore

$$\int_t^x K(x,s)K(s,t)ds=\int_t^x e^s e^t ds=e^{x+t}-e^{2t}$$

In the interval $(x,1)$ we have $s>x>t$, and therefore

$$\int_x^1 K(x,s)K(s,t)ds=\int_x^1 e^x e^t ds=(1-x)e^{x+t}$$

Adding the integrals thus found, we obtain

$$K_2(x,t)=(2-x)e^{x+t}-\frac{1+e^{2t}}{2}\quad (x>t)$$

We will find the expression for $K_2(x,t)$ for $x<t$ if we interchange the arguments x and t in the expression $K_2(x,t)$ for $x>t$:

$$K_2(x,t)=(2-x)e^{x+t}-\frac{1+e^{2t}}{2}\quad (x<t)$$

Therefore, the second iterated kernel is of the form

$$K_2(x,t)=\begin{cases}(2-t)e^{x+t}-\dfrac{1+e^{2x}}{2}, & if\ 0\le x\le t,\\ (2-x)e^{x+t}-\dfrac{1+e^{2t}}{2}, & if\ t\le x\le 1.\end{cases}$$

10.4.2.1 Note: If the kernel $K(x,t)$, which is specified in the square $a\le x\le b,\ a\le t\le b$ by various analytic expressions, is not symmetric, then one should consider the case $x<t$ separately.

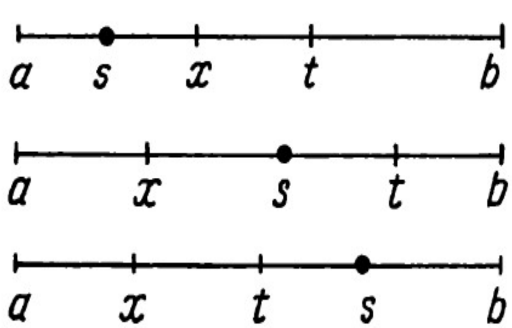

Fig. 3

From Fig. 3 we can observe the case $x < t$ and we can compute $K_2(x,t)$ as

$$K_2(x,t) = \int_a^b K(x,s)K(s,t)ds = \int_a^x + \int_x^t + \int_t^b .$$

10.4.3 Example: Find the iterated kernels $K_1(x,t)$ and $K_2(x,t)$ when $a = 0,\ b = 1$ and

$$K(x,t) = \begin{cases} x + t, & if\ 0 \le x < t, \\ x - t, & if\ t < x \le 1 \end{cases}.$$

Solution. We have $K_1(x,t) = K(x,t)$,

$$K_2(x,t) = \int_0^1 K(x,s)K(s,t)ds,$$

where,

$$K(x,s) = \begin{cases} x + s, & if\ 0 \le x < s, \\ x - s, & if\ s < x \le 1, \end{cases} \quad K(s,t) = \begin{cases} s + t, & if\ 0 \le s < t, \\ s - t, & if\ t < s \le 1. \end{cases}$$

Since the given kernel $K(x,t)$ is not symmetric, we consider two cases separately when finding $K_2(x,t)$: (1) $x < t$ and (2) $x > t$.

(1) For first case: Let $x < t$. Then from Fig. 3

$$K_2(x,t) = I_1 + I_2 + I_{3.}$$

Where,

$$I_1 = \int_0^x (x - s)(s + t)ds = \frac{x^3}{6} + \frac{x^2 t}{2},$$

$$I_2 = \int_x^t (x + s)(s + t)ds = \frac{5t^3}{6} - \frac{5x^3}{6} + \frac{3xt^2}{2} - \frac{3x^2 t}{2},$$

$$I_3 = \int_t^1 (x + s)(s - t)ds = \frac{t^3}{6} + \frac{xt^2}{2} - xt + \frac{x}{2} - \frac{t}{2} + \frac{1}{3}.$$

Adding these integrals, we obtain

$$K_2(x,t) = t^3 - \frac{2}{3}x^3 - x^2 t + 2xt^2 - xt + \frac{x - t}{2} + \frac{1}{3} \quad (x < t).$$

(2) For first case: Let $x > t$, from Fig. 2

$$K_2(x,t) = I_1 + I_2 + I_3.$$

Where

$$I_1 = \int_0^t (x-s)(s+t)ds = \frac{3xt^2}{2} - \frac{5t^3}{6},$$

$$I_2 = \int_t^x (x-s)(s-t)ds = \frac{x^3}{6} - \frac{t^3}{6} - \frac{x^2 t}{2} + \frac{xt^2}{2},$$

$$I_3 = \int_x^1 (x+s)(s-t)ds = -\frac{5x^3}{6} + \frac{3x^2 t}{2} + \frac{x-t}{2} - xt + \frac{1}{3}.$$

Adding these integ1als, we obtain

$$K_2(x,t) = -\frac{2}{3}x^3 - t^3 + x^2 t + 2xt^2 - xt + \frac{x-t}{2} + \frac{1}{3} \quad (x > t).$$

Therefore, the second iterated kernel is of the form

$$K_2(x,t) = \begin{cases} -\frac{2}{3}x^3 + t^3 - x^2 t + 2xt^2 - xt + \frac{x-t}{2} + \frac{1}{3} & if\ 0 \le x < t), \\ -\frac{2}{3}x^3 - t^3 + x^2 t + 2xt^2 - xt + \frac{x-t}{2} + \frac{1}{3}, & if\ t < x \le 1. \end{cases}$$

The other iterated kernels $K_n(x,t)$ $(n = 3, 4, \dots)$ are found in similar fashion.

10.4.4 Example: Find the iterated kernels for the kernel $K(x,t) = xe^t$;

where $a = 0,\ b = 1.$

Solution. By using formulas (10.3), we find iterated kernels as

$K_1(x,t) = xe^t,$

$$K_2(x,t) = \int_0^1 (xe^s)\,(se^t)ds = xe^t,$$

$$K_3(x,t) = \int_0^1 (xe^s)\,(se^t)ds = xe^t,$$

$$K_4(x,t) = \int_0^1 (xe^s)\,(se^t)ds = xe^t,$$

$$K_5(x,t) = \int_0^1 (xe^s)\,(se^t)ds = xe^t,$$

From this it follows that iterated kernels are of the form:

$$K_n(x,t) = \int_0^1 (xe^s)\,(se^t)ds = xe^t,$$

where $n = 1, 2, 3, \dots$

10.4.5 We now present an example of how to construct the resolvent kernel of an integral equation using iterated kernels. Consider the following integral equation:

$$\varphi(x) - \lambda \int_0^1 xt\varphi(t)dt = f(x) \qquad (10.11)$$

Here $K(x,t) = xt;\ a = 0,\ b = 1$**.** We iteratively find

$$K_1(x,t) = xt,$$

$$K_2(x,t) = \int_0^1 (xz)(zt)dz = \frac{xt}{3},$$

$$K_3(x,t) = \frac{1}{3}\int_0^1 (xz)(zt)dz = \frac{xt}{3^2},$$

$$\dots\dots\dots\dots.$$

$$K_n(x,t) = \frac{xt}{3^{n-1}},$$

According to formula (10.5)

$$R(x,t;\ \lambda) = \sum_{n=1}^{\infty} K_n(x,t)\lambda^{n-1} = xt\sum_{n=1}^{\infty}\left(\frac{\lambda}{3}\right)^{n-1} = \frac{3xt}{3-\lambda}$$

where $|\lambda| < 3$.

By applying the formula (10.7) the solution of the integral equation (10.11) will be written as

$$\varphi(x) = f(x) + \lambda\int_0^1 \frac{3xt}{3-\lambda} f(t)dt$$

In particular, for $f(x) = x$ we get

$$\varphi(x) = \frac{3x}{3-\lambda} \text{ where } \lambda \neq 3.$$

10.4.6 Example: Construct resolvent kernels for the kernel $K(x,t) = e^{x+t}$ for $a = 0$ and $b = 1$.

Solution. We have $K(x,t) = e^{x+t};\ a = 0,\ b = 1$**.** We iteratively find

$$K_1(x,t) = e^{x+t},$$

$$K_2(x,t) = \int_0^1 (e^{x+z})(e^{z+t})dz = e^{x+t}\frac{e^2-1}{2},$$

$$K_3(x,t) = \int_0^1 \left(\frac{e^2-1}{2}e^{x+z}\right)(e^{z+t})dz = \frac{(e^2-1)^2}{2^2}e^{x+t}$$

… … … … .

$$K_n(x,t) = \frac{(e^2-1)^{n-1}}{2^{n-1}}e^{x+t},$$

According to formula (10.5)

$$R(x,t;\ \lambda) = \sum_{n=1}^{\infty} K_n(x,t)\lambda^{n-1} = e^{x+t}\sum_{n=1}^{\infty}\left(\frac{e^2-1}{2}\lambda\right)^{n-1} = \frac{2e^{x+t}}{2-(e^2-1)\lambda}.$$

This series converges when $\left|\frac{e^2-1}{2}\lambda\right| < 1,$

i.e., $|\lambda| < \frac{2}{e^2-1}$.

10.4.7 Constructing Resolvent Kernels for Two Orthogonal Kernels

We now construct the resolvent kernels to the orthogonal kernels as follows:

If $M(x,t)$ and $N(x,t)$ are two orthogonal kernels, then the resolvent kernel $R\ (x,t;\ \lambda)$ corresponding to the kernel $K(x,t) = M + N$, is equal to the sum of the resolvent kernels $R_1(x,t;\ \lambda)$ and $R_2(x,t;\ \lambda)$ which correspond to each of these kernels. Consider the following example:

10.4.7.1 Example: Find the resolvent kernel for the kernel $K(x,t) = xt + x^2t^2,\ a = -1,$ $b = 1$.

Solution. As was shown above, the kernels $M(x,t) = xt$ and $N(x,t) = x^2t^2$ are orthogonal on [-1, 1]. For this reason, the resolvent kernel of the kernel $K(x,t)$ is equal to the sum of the resolvent kernels of the kernels $M(x,t)$ and $N(x,t)$. Utilizing the results of problems 4 and 5 (in Exercise (10b)), we obtain

$$R_K(x,t;\ \lambda) = R_M(x,t;\ \lambda) + R_N(x,t;\ \lambda) = \frac{3xt}{3-2\lambda} + \frac{5x^2t^2}{5-2\lambda}$$

where $|\lambda| < \frac{3}{2}$.

10.4.7.2 Notes:

1. Even though we can construct resolvent kernels for pairwise orthogonal as follows:

If the kernels $M^{(1)}(x,t), M^{(2)}(x,t),\dots, M^{(n)}(x,t)$ are pairwise orthogonal, then the resolvent kernel corresponding to their sum,

$$K(x,t) = \sum_{m=1}^{n} M^{(m)}(x,t)$$

is equal to the sum of the resolvent kernels corresponding to each of the terms.

2. Let us use the term ``n^{th} trace$''$ of the kernel $K(x,t)$ for the quantity

$$A_n = \int_a^b K_n(x,x)dx, (n = 1, 2, \dots) \quad (10.12)$$

Where $K_n(x,x)$ is the n^{th} iterated kernel for the kernel $K(x,t)$.

3. The following formula holds for the Fredholm determinant $D(\lambda)$:

$$\frac{D'(\lambda)}{D(\lambda)} = -\sum_{n=1}^{\infty} A_n \lambda^{n-1} \quad (10.13)$$

The radius of convergence of the power series (10.13) is equal to the smallest of the moduli of the characteristic numbers.

10.5 SUMMARY:

This lesson introduces iterated kernels as a fundamental tool for solving Fredholm integral equations of the second kind. It describes how these kernels are systematically constructed from the original kernel and how they contribute to the Neumann series representation of the resolvent kernel. Conditions for convergence of the series are discussed, and several illustrative examples are presented. The role of kernel symmetry and orthogonality in simplifying computation is also explored.

10.6 TECHNICAL TERMS:

- **Iterated Kernel:** Functions generated recursively from an initial kernel, used in constructing the resolvent kernel.
- **Resolvent Kernel:** A kernel expressed as a series involving iterated kernels, which is used to find the solution of a Fredholm equation.
- **Neumann Series:** An infinite series used to represent the resolvent kernel; its convergence is essential for the validity of the solution.
- **Orthogonal Kernels:** Kernels whose mixed integrals vanish, simplifying the computation of the resolvent kernel.

10.7 SELF-ASSESSMENT QUESTIONS:

Exercise (10a): Find the iterated kernels of the following kernels for specified *a* and *b.*

1. $K(x,t) = x - t;\ a = -1,\ b = 1.$

2. $K(x,t) = \sin(x-t)\,;\ a = 0,\ b = \frac{n}{2}\ (n = 2,3).$
3. $K(x,t) = (x-t)^2;\ a = -1,\ b = 1\ (n = 2,3).$
4. $K(x,t) = x + \sin t\,;\ a = -\pi,\ b = \pi.$
5. $K(x,t) = e^x cost;\ a = 0,\ b = \pi.$

In the following problems, find $K_2(x,t)$:

6. $K(x,t) = e^{|x-t|};\ a = 0,\ b = 1.$
7. $K(x,t) = e^{|x|+t};\ a = -1,\ b = 1.$

Exercise (10b): Construct resolvent kernels for the following kernels:

1. $K(x,t) = \sin x\, cost;$ $a = 0,\ b = \frac{\pi}{2}.$
2. $K(x,t) = xe^t;$ $a = -1,\ b = 1.$
3. $K(x,t) = (1+x)(1-t);$ $a = -1,\ b = 0.$
4. $K(x,t) = x^2t^2;$ $a = -1,\ b = 1.$
5. $K(x,t) = xt;$ $a = -1,\ b = 1.$

Exercise (10c): Find resolvent kernels for the following kernels:

1. $K(x,t) = \sin x\, cost + cos2xsin2t;\ a = 0,\ b = 2\pi.$
2. $K(x,t) = 1 + (2x-1)(2t-1);\ a = 0,\ b = 1.$

Exercise (10d):

1. Show that for the Volterra equation

$$\varphi(x) - \lambda \int_0^x K(x,t)\varphi(t)dt = f(x)$$

the Fredholm determinant $D(\lambda) = e^{-A_1\lambda}$ and, consequently, the resolvent kernel for the Volterra equation is an entire analytic function of λ.

2. Let $R(x,t;\ \lambda)$ be the resolvent kernel for some kernel $K(x,t)$.

Show that the resolvent kernel of the equation

$$\varphi(x) - \mu \int_a^b R(x,t;\ \lambda)\varphi(t)dt = f(x)$$

is equal to $R(x,t;\ \lambda+\mu)$

3. Let

$$\int_a^b \int_a^b K^2(x,t)dxdt = B^2$$

$$\int_a^b \int_a^b K_n^2\,(x,t)dxdt = B_n^2$$

Where $K_n(x,t)$ is the n^{th}iterated kernel for the kernel $K(x,t)$. Prove that if $B_2 = B_n^2$, then for any *n* we will have $B_n = B^n$.

SELF-ASSESSMENT ANSWERS:

Exercise (10a)

1. $K_{2n-1}(x,t) = \left(-\frac{4}{3}\right)^{n-1}(x-t),$

   $$K_{2n}(x,t) = 2(-1)^n\left(\frac{4}{3}\right)^{n-1}\left(xt+\frac{1}{3}\right), (n = 1,2,3,\ldots).$$

2. $K_2(x,t) = \frac{\sin(x+t)}{2} - \frac{\pi}{4}\cos(x-t),$

   $$K_3(x,t) = \frac{4-\pi 2}{16} sin(x-t.)$$

3. $K_2(x,t) = \frac{2}{3}(x+t)^2 + 2x^2t^2 + \frac{4}{3}xt + \frac{2}{5},$

   $$K_3(x,t) = \frac{56}{15}(x^2+t^2) + \frac{8}{3}x^2t^2 - \frac{32}{9}xt + \frac{8}{15}.$$

4. $K_{2n-1}(x,t) = (2\pi)^{2n-2}(x + sint)$

   $$K_{2n}(x,t) = (2\pi)^{2n-1}(1 + \mathrm{x}\sin t), (n = 1,2,\ldots)$$

5. $K_n(x,t) = (-1)^{n-1}\left(\frac{e^{\pi}+1}{2}\right)^{n-1} e^x cost$

6. $K_2(x,t)\begin{cases} \frac{e^{x+t}+e^{2-x-t}}{2} + (t-x-1)e^{t-x}, 0 \le x \le t \\ \frac{e^{x+t}+e^{2-x-t}}{2} + (x-t-1)e^{x-t}, t \le x \le 1 \end{cases}$

7. $K_2(x,t)\begin{cases} \frac{e^2+1}{2}e^{t-x}, -1 \le x \le 0 \\ \frac{e^2+1}{2}e^{t+x}, 0 \le x \le 1 \end{cases}$

Exercise (10b):

1. $R(x,t;\lambda) = \frac{2sinx\ cost}{2-\lambda};\ |\lambda| < 2$

2. $R(x,t;\lambda) = \frac{xe^{t+1}}{e-2\lambda}; |\lambda| < \frac{e}{2}$
3. $R(x,t;\lambda) = \frac{3(1+x)(1-t)}{3-2\lambda}; |\lambda| < \frac{2}{3}$
4. $R(x,t;\lambda) = \frac{5x^2t^2}{5-2\lambda}; \ |\lambda| < \frac{5}{2}$
5. $R(x,t;\lambda) = \frac{3xt}{3-2\lambda}; \ |\lambda| < \frac{3}{2}$

Exercise (10c):

1. $R(x,t;\lambda) = sinx\ cost + cos\ 2x\ sin\ 2t$
2. $R(x,t;\lambda) = \frac{1}{1-\lambda} + \frac{3(2x-1)(2t-1)}{3-\lambda}; |\lambda| < 1$

10.8 SUGGESTED READINGS:

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3. Rainer Kress, *Linear Integral Equations*, Springer, 2014 (Third Edition). ISBN-978-1447171474.
4. I. G. Petrovsky, *Lectures on Partial Differential Equations*, Dover Publications, 2012. ISBN-978-0486659640.
5. M. A. Krasnoselskii, *Integral Equations of the First Kind: Invariant Imbedding Method and Applications*, CRC Press, 1994. ISBN-978-2884490651.

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- **Prof. P. Vijaya Laxmi**

LESSON- 11
INTEGRAL EQUATIONS WITH DEGENERATE KERNELS

OBJECTIVE:

- To understand the fundamental concepts of degenerate kernels.
- To use degenerate kernels to simplify integral equation making it easier to analyse and solve.
- To provide several examples to understand the solving of integral equations in different ways

STRUCTURE:

11.1 Introduction

11.2 Formation of Integral Equations with Degenerate Kernels

11.3 Hammerstein-Type Equation

11.4 Examples

11.5 Summary

11.6 Technical Terms

11.7 Self-Assessment Questions

11.8 Suggested Readings

11.1 INTRODUCTION:

Integral equations with degenerate kernels are a specific type where the kernel function simplifies to a form that allows the integral equation to be reduced to a finite system of linear equations. This reduction is possible because the kernel can be written as a finite sum of products of functions, each depending on only one variable. Such kernels make the analysis and solution of the equation more tractable and are often used to approximate more complicated kernels in both theoretical and practical problems.

11.2 FORMATION OF INTEGRAL EQUATIONS WITH DEGENERATE KERNELS:

We define degenerate kernels and formulate the integral equation using degenerate kernels.

11.2.1 Definition: The kernel $K(x,t)$ of a Fredholm integral equation of the second kind is called *degenerate* if it is the sum of a finite number of products of functions of *x* alone by functions of *t* alone;

i.e., if it is of the form

$$K(x,t) = \sum_{k=1}^{n} a_k(x) b_k(t). \qquad (11.1)$$

We shall consider the functions $a_k(x)$ and $b_k(t)$ $(k = 1, 2, \dots, n)$ continuous in the basic square $a \le x, t \le b$ and linearly independent.

11.2.2 Note:

- The integral equation with degenerate kernel (11.1)

$$\varphi(x) - \lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x) b_k(t)\right] \varphi(t) dt = f(x) \tag{11.2}$$

is solved in the following manner:

Rewrite (11.2) as

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} a_k(x) \int_a^b b_k(t)\varphi(t) dt \tag{11.3}$$

and introduce the notation

$$\int_a^b b_k(t)\varphi(t) dt = C_k \ , 1 \le k \le n. \tag{11.4}$$

Then (11.3) becomes

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x) \tag{11.5}$$

where C_k are unknown constants, since the function $\varphi(x)$ is unknown.

Thus, the solution of an integral equation with degenerate kernel reduces to finding the constants C_k $(k = 1, 2, \dots, n)$. Putting the expression (11.5) into the integral equation (11.2), we get

$$\sum_{m=1}^{n} \left\{ C_m - \int_a^b b_m(t) \left[f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t) \right] dt \right\} a_m(x) = 0$$

Whence it follows, by virtue of the linear independence of the functions $a_m(x)$ $(m = 1, 2, \dots, n)$ that

$$C_m - \int_a^b b_m(t) \left[f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t) \right] dt = 0$$

or

$$C_m - \lambda \sum_{k=1}^{n} C_k \int_a^b a_k(t) b_m(t) dt = \int_a^b b_m(t) f(t) dt \quad (m = 1, 2, \ldots, n)$$

For the sake of brevity, we introduce the notations

$$a_{km} = \int_a^b a_k(t) b_m(t) dt \,, f_m = \int_a^b b_m(t) f(t) dt$$

and find that

$$C_m - \lambda \sum_{k=1}^{n} a_{km} C_k = f_m \quad (m = 1, 2, \ldots, n)$$

or, in expanded form,

$$\begin{array}{c} (1-\lambda a_{11})C_1 - \lambda a_{12} C_2 - \cdots - \lambda a_{1n} C_n = f_{1,} \\ -\lambda a_{21} C_1 + (1-\lambda a_{22}) C_2 - \cdots - \lambda a_{2n} C_n = f_2 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \\ -\lambda a_{n1} C_1 - \lambda a_{n2} C_2 - \cdots + (1-\lambda a_{nn}) C_n = f_n \end{array} \quad (11.6$$

For finding the unknowns C_k we have a linear system of *n* algebraic equations in *n* unknowns. The determinant of this system is

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & 0-\lambda a_{12} & \cdots & 0-\lambda a_{1n} \\ 0-\lambda a_{21} & 1-\lambda a_{22} & \cdots & 0-\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0-\lambda a_{n1} & 0-\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \quad (11.7$$

If $\Delta(\lambda) \neq 0$ then the system (11.6) has a unique solution $C_1, C_2, \ldots, C_n$ which is obtained from Cramer's formulas

$$C_k = \frac{1}{\Delta(\lambda)} \begin{vmatrix} 1-\lambda a_{11} & \cdots & -\lambda a_{1k-1} f_1 - \lambda a_{1k+1} & \cdots & 0-\lambda a_{1n} \\ 0-\lambda a_{21} & \cdots & -\lambda a_{2k-1} f_2 - \lambda a_{2k+1} & \cdots & 0-\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots & 0-\lambda a_{2n} \\ 0-\lambda a_{n1} & \cdots & -\lambda a_{nk-1} f_n - \lambda a_{nk+} & \cdots & 1-\lambda a_{nn} \end{vmatrix} \quad (11.8)$$

where $(k = 1, 2, \ldots, n)$

The solution of the integral equation (11.2) is the function $\varphi(x)$ defined by the equality

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x)$$

where the coefficients C_k $(k = 1, 2, \ldots, n)$ are determine from formulas (11.8).

- The system (11.6) may be obtained if both sides of (11.5) are consecutively multiplied by $a_1(x)$, $a_2(x), \ldots, a_n(x)$ and integrated from *a* to *b* or if we put (11.5) into (11.4) for $\varphi(x)$, replacing *x* by *t*.

11.2.3 Example: Solve the integral equation

$$\varphi(x) - \lambda \int_{-\pi}^{\pi} (x\cos t + t^2 \sin x + \cos x \sin t)\varphi(t)dt = x \qquad (11.9)$$

Solution. Write the equation in the following form:

$$\varphi(x) = \lambda x \int_{-\pi}^{\pi} \varphi(t)\cos t dt + \lambda \sin x \int_{-\pi}^{\pi} t^2\varphi(t)dt + \lambda \cos x \int_{-\pi}^{\pi} \varphi(t) \sin t\, dt + x$$

We introduce the notations

$$C_1 = \int_{-\pi}^{\pi} \varphi(t) \cos t\, dt;\; C_2 = \int_{-\pi}^{\pi} t^2\varphi(t)dt;\; C_3 = \int_{-\pi}^{\pi} \varphi(t) \sin t\, dt \qquad (11.10)$$

where C_1 C_2, C_3 are unknown constants. Then equation (11.9) assumes the form

$$\varphi(x) = C_1\lambda x + C_2\lambda \sin x + C_3\lambda \cos x + x \qquad (11.11)$$

Substituting expression (11.11) into (11.10), we get

$$C_1 = \int_{-\pi}^{\pi} (C_1\lambda t + C_2\lambda \sin t + C_3\lambda \cos t + t) \cos t\, dt,$$

$$C_2 = \int_{-\pi}^{\pi} (C_1\lambda t + C_2\lambda \sin t + C_3\lambda \cos t + t)t^2 dt,$$

$$C_3 = \int_{-\pi}^{\pi} (C_1\lambda t + C_2\lambda \sin t + C_3\lambda \cos t + t) \sin t\, dt$$

or

$$C_1\left(1 - \lambda \int_{-\pi}^{\pi} t\cos t\, dt\right) - C_2\lambda \int_{-\pi}^{\pi} \sin t \cos t\, dt - C_3\lambda \int_{-\pi}^{\pi} \cos^2 t\, dt = \int_{-\pi}^{\pi} t\cos t dt$$

$$-C_1\lambda \int_{-\pi}^{\pi} t^3 dt + C_2\left(1 - \lambda \int_{-\pi}^{\pi} t^2 \sin t\, dt\right) - C_3\lambda \int_{-\pi}^{\pi} t^2 \cos t\, dt = \int_{-\pi}^{\pi} t^3 dt,$$

$$-C_1\lambda \int_{-\pi}^{\pi} t\sin t\, dt - C_2\lambda \int_{-\pi}^{\pi} \sin^2 t\, dt + C_3\left(1 - \lambda \int_{-\pi}^{\pi} \cos t \sin t\, dt\right) = \int_{-\pi}^{\pi} t \sin t\, dt$$

By evaluating the integrals that enter into this system we obtain a system of algebraic equations for finding the unknowns C_1, C_2, C_3:

$$\begin{aligned} C_1 - \lambda\pi C_3 &= 0 \\ C_2 + 4\lambda\pi C_3 &= 0 \\ -2\lambda\pi C_1 - \lambda\pi C_2 + C_3 &= 2\pi \end{aligned} \tag{11.12}$$

The determinant of this system is

$$\Delta(\lambda) = \begin{vmatrix} 1 & 0 & -\lambda\pi \\ 0 & 1 & 4\lambda\pi \\ -2\lambda\pi & -\lambda\pi & 1 \end{vmatrix} = 1 + 2\lambda^2\pi^2 \neq 0$$

The system (11.12) has a unique solution

$$C_1 = \frac{2\pi^2\lambda}{1+2\lambda^2\pi^2};\ C_2 = -\frac{8\pi^2\lambda}{1+2\lambda^2\pi^2};\ C_3 = \frac{2\pi}{1+2\lambda^2\pi^2}$$

Substituting the values of C_1, C_2, C_3 thus found into (11.11), we obtain the solution of the given integral equation

$$\varphi(x) = \frac{2\pi\lambda}{1+2\lambda^2\pi^2}(\pi\lambda x - 4\pi\lambda \sin x + \cos x) + x.$$

11.3 HAMMERSTEIN-TYPE EQUATION:

The canonical form of the *Hammerstein-type equation* is

$$\varphi(x) = \int_a^b \mathrm{K}(x,t)\, f(t, \varphi(t))dt \tag{11.13}$$

where $K(x,t), f(t,u)$ are given functions and $\varphi(x)$ is the unknown function.

The following equations readily reduce to equations of type (11.13):

$$\varphi(x) = \int_a^b \mathrm{K}(x,t)\, f\big(t, \varphi(t)\big)dt + \psi(x) \tag{11.14}$$

where $\psi(x)$ is the known function, so that the difference between homogeneous and nonhomogeneous equations, which is important in the linear case, is almost of no importance in the nonlinear case. We shall call the function $K(x,t)$ the kernel of equation (11.13).

11.3.1 Note: Let $K(x,t)$ be a degenerate kernel, i. e.,

$$K(x,t) = \sum_{i=1}^{m} a_i(x) b_i(t) \tag{11.1:}$$

Then equation (11.13**)** takes the form

$$\varphi(x) = \sum_{i=1}^{m} a_i(x) \int_a^b b_i(t) f(t, \varphi(t))dt \tag{11.1(}$$

Put

$$C_i = \int_a^b b_i(t) f\big(t, \varphi(t)\big) dt \quad (i = 1, 2, \dots, m) \qquad (11.17)$$

where the C_i are as yet unknown constants. Then, by virtue of (11.16), we will have

$$\varphi(x) = \sum_{i=1}^{m} C_i a_i(x) \qquad (11.18)$$

Substituting into (11.17) the expression (11.18) for $\varphi(x)$, we get m equations (generally, transcendental) containing m unknown quantities $C_1, C_2, \dots, C_m$:

$$C_i = \psi_i(C_1, C_2, \dots, C_m) \;\; (i = 1, 2, \dots, m) \qquad (11.19)$$

When $f(t, u)$ is a polynomial in u,

$$f(t, u) = p_0(t) + p_1(t)u + \dots + p_n(t)u^n \qquad (11.20)$$

where $p_0(t),\; p_1(t), \dots, p_n(t)$ are, for instance, continuous functions of t on the interval $[a, b]$, the system (11.19) is transformed into a system of algebraic equations in $C_1, C_2, \dots, C_m$. If there exists a solution of the system (11.19), that is, if there exist m numbers

$$C_1^0, C_2^0, \dots, C_m^0$$

such that their substitution into (11.19) reduces the equations to identities, then there exists a solution of the integral equation (11.16) defined by the equality (11.18):

$$\varphi(x) = \sum_{i=1}^{m} C_i^0 a_i(x)$$

It is obvious that the number of solutions (generally, complex) of the integral equation (11.16) is equal to the number of solutions -of the system (11.19).

11.3.2 Solve the integral equation

$$\varphi(x) = \lambda \int_0^1 xt\varphi^2(t) dt \qquad (11.21$$

where λ is a parameter.

Solution. Put

$$C = \int_0^1 t\varphi^2(t) dt \qquad (11.22$$

Then

$$\varphi(x) = C\lambda x \qquad (11.23)$$

Substituting $\varphi(x)$ in the form (11.23) into the relation (11.22), we get

$$C = \int_0^1 t\lambda^2 C^2 t^2 dt$$

Whence

$$C = \frac{\lambda^2}{4} C^2 \qquad (11.24)$$

Equation (11.24) has two solutions

$$C_1 = 0, \qquad C_2 = \frac{4}{\lambda^2}$$

Consequently, integral equation (11.21) also has two solutions for any $\lambda \neq 0$:

$$\varphi_1(x) \equiv 0, \qquad \varphi_2(x) = \frac{4}{\lambda} x$$

There exist simple nonlinear integral equations which do not have real solutions at all.

Consider, for example, the equation

$$\varphi(x) = \frac{1}{2}\int_0^1 e^{\frac{x+t}{2}} \left(1 + \varphi^2(t)\right) dt \qquad (11.25)$$

Put

$$C = \frac{1}{2}\int_0^1 e^{\frac{t}{2}} \left(1 + \varphi^2(t)\right) dt \qquad (11.26)$$

Then

$$\varphi(x) = Ce^{\frac{x}{2}} \qquad (11.27)$$

For a determination of the constant C, we obtain the equation

$$\left(e^{\frac{3}{2}} - 1\right) C^2 - 3C + 3\left(e^{\frac{1}{2}} - 1\right) = 0 \qquad (11.28)$$

It is easy to verify that equation (11.28) does not have real roots and, hence, the integral equation (11.25) has no real solutions.
On the other hand, let us consider the equation

$$\varphi(x) = \int_0^1 a(x)a(t)\varphi(t)\sin\left(\frac{\varphi(t)}{a(t)}\right) dt \qquad (11.29)$$

($a(t) > 0$ for all $t \in [0, 1]$)

In order to determine the constant C, we arrive at the equation

$$1 = \int_0^1 a^2(t)dt.\sin C \tag{11.30}$$

If

$$\int_0^1 a^2(t)dt > 1,$$

then equation (11.30) and, hence, the original integral equation (11.29) as well, has an infinite number of real solutions.

11.4 EXAMPLES:

We can explore several illustrative examples that demonstrate the methods and solution strategies in detail. These examples provide deeper insight into solving integral equations, especially those involving degenerate kernels and nonlinear terms.

11.4.1 Solve the given integral equation with degenerate kernels

$$(x) - 4\int_0^{\frac{\pi}{2}} \sin^2 x\varphi(t)\, dt = 2x - \pi \tag{11.31}$$

Solution. Write the equation in the following form:

$$\varphi(x) = 4\sin^2 x \int_0^{\frac{\pi}{2}} \varphi(t)\, dt + 2x - \pi$$

Where

$$C_1 = \int_0^{\frac{\pi}{2}} \varphi(t)dt \tag{11.32}$$

where C_1 is unknown constants. Then equation (11.31) assumes the form

$$\varphi(x) = 4\sin^2 x\, C_1 + 2x - \pi \tag{11.33}$$

Substituting expression (11.33) into (11.32), we get

$$C_1 = \int_0^{\frac{\pi}{2}} (4C_1 \sin^2 t + 2t - \pi)\, dt$$

$$C_1\left(1-4\int_0^{\frac{\pi}{2}}\sin^2 t\ dt\right)=\int_0^{\frac{\pi}{2}}(2t-\pi)dt$$

$$C_1(1-\pi)=\frac{-\pi^2}{4}$$

Thus,

$$C_1=\frac{\pi^2}{4(\pi-1)} \tag{11.34}$$

Substituting the value of $C_1,$ thus found into (11.33), we obtain the solution of the given integral equation

$$\varphi(x)=\frac{\pi^2}{\pi-1}\sin^2 x+2x-\pi$$

11.4.2 Solve the given integral equation with degenerate kernels

$$\varphi(x)-\lambda\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\ t\,\varphi(t)\ dt=\cot x \tag{11.35}$$

Solution. Write the equation in the following form:

Where

$$\varphi(x)=\lambda\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\ t\,\varphi(t)\ dt+\cot x$$

Take,

$$C_1=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\ t\,\varphi(t)\ dt \qquad (11.36$$

where C_1 is unknown constants. Then equation (11.35) assumes the form

$$\varphi(x)=\lambda C_1+\cot x \tag{11.37}$$

Substituting expression (11.37) into (11.36), we get

$$C_1=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan t\,(\lambda C_1+\cot t)$$

$$C_1\left(1-\lambda\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\, t\ \ dt\right) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan\, t\ \cot t\, dt$$

Thus,

$$C_1 = \frac{\pi}{2} \tag{11.38}$$

Substituting the value of C_1, thus found into (11.37), we obtain the solution of the given integral equation

$$\varphi(x) = \lambda\frac{\pi}{2} + \cot x.$$

11.4.3 Solve the integral equation

$$\varphi(x) = 2\int_0^1 xt\ \varphi^3(t)\ dt \tag{11.39}$$

Solution.

$$C = \int_0^1 t\varphi^3(t)dt \tag{11.40}$$

Then

$$\varphi(x) = 2Cx \tag{11.41}$$

Substituting $\varphi(x)$ in the form (11.41) into the relation (11.40), we get

$$C = 8C^3\int_0^1 t^4 dt$$

Whence

$$C = \frac{8}{5}C^3 \tag{11.42}$$

Equation (11.42) has two solutions

$$C_1 = 0, \qquad C_{2,3} = \pm\sqrt{\frac{5}{8}}$$

Consequently, integral equation (11.39) also has two solutions for any

$$\varphi_1(x) \equiv 0, \qquad \varphi_{2,3}(x) = \pm\sqrt{\frac{5}{2}}x$$

11.4.4 Solve the integral equation

$$\varphi(x) = \int_0^1 (1 + \varphi^2(t))dt \tag{11.43}$$

Solution.

$$C = \int_0^1 (1 + \varphi^2(t))dt \tag{11.44}$$

Then

$$\varphi(x) = C \tag{11.45}$$

Substituting $\varphi(x)$ in the form (11.45) into the relation (11.44), we get

$$C = \int_0^1 (1 + C^2)dt$$

hence

$$C = 1 + C^2 \tag{11.46}$$

Which implies no real solutions.

11.5 SUMMARY:

Integral equations with degenerate kernels are a special class of integral equations where the kernel can be expressed as a finite sum of separable functions. This simplification allows the integral equation to be reduced to a system of linear equations, making it much easier to solve analytically or numerically. These equations often arise in physics and engineering problems, particularly in systems with symmetric or structured interactions.

11.6 TECHNICAL TERMS:

Degenerate of the Kernel:

The kernel $K(x,t)$ is called degenerate if it can be written as a finite sum of number of products of functions of x alone by functions of t alone;
i.e., if it is of the form

$$K(x,t) = \sum_{k=1}^{a} a_k(x)b_k(t)$$

Hammerstein-Type Equation:

A Hammerstein type equation is a kind of nonlinear integral equation that typically has the form:

$$\varphi(x) = \int_a^b \mathrm{K}(x,t)\, f(t,\varphi(t))dt$$

where $K(x,t), f(t,u)$ are given functions and $\varphi(x)$ is the unknown function.

11.7 SELF-ASSESSMENT QUESTIONS:

Exercise (11a): Solve the following integral equations with degenerate kernels:

1. $\varphi(x) - \int_{-1}^{1} e^{\arcsin x}\, \varphi(t)dt = \tan x.$
2. $\varphi(x) - \lambda \int_0^1 \cos(q\, ln\, t)\, \varphi(t)dt = 1.$
3. $\varphi(x) - \lambda \int_0^1 arc\cos t\, \varphi(t)dt = \frac{1}{\sqrt{1-x^2}}.$
4. $\varphi(x) - \lambda \int_0^1 \left(\ln\frac{1}{t}\right)^p \varphi(t)dt = 1 \;\; (p > -1).$
5. $\varphi(x) - \lambda \int_0^1 (x\, ln\, t - t\, ln\, x)\, \varphi(t)dt = \frac{6}{5}(1-4x).$
6. $\varphi(x) - \lambda \int_0^{\frac{\pi}{2}} \sin x \cos t\, \varphi(t)dt = \sin x.$
7. $\varphi(x) - \lambda \int_0^{2\pi} |\pi - t| \sin x\, \varphi(t)dt = x.$
8. $\varphi(x) - \lambda \int_0^{\pi} \sin(x-t)\, \varphi(t)dt = \cos x.$
9. $\varphi(x) - \int_0^{2\pi} (\sin x \cos t - \sin 2x \cos 2t + \sin 3x\, cos\, 3t)\, \varphi(t)dt = \cos x.$
10. $\varphi(x) - \frac{1}{2}\int_{-1}^{1} \left|x - \frac{1}{2}(3t^2-1)\right| + \frac{1}{2}t(3x^2-1)\, \varphi(t)dt = 1.$

Exercise (11b): Solve the following integral equations:

1. $\varphi(x) = \int_{-1}^{1} (xt + x^2t^2)\varphi^2(t)dt.$
2. $\varphi(x) = \int_0^1 x^2t^2\varphi^3(t)dt.$
3. $\varphi(x) = \int_{-1}^{1} \frac{xt}{1+\varphi^2(t)}dt.$
4. Show that the integral equation

$$\varphi(x) = \frac{1}{2}\int_0^1 a(x)a(t)\big(1+\varphi^2(t)\big)\, dt$$

$$(a(x) > 0 \text{ for all } x \in [0,1])$$

has no real solutions if $\int_0^1 a^2(x)dx > 1.$

Self-Assessment Answers:

Exercise (11a)

1. $\varphi(x) = \tan x.$
2. $\varphi(x) = \frac{1+q^2}{1+q^2-\lambda}.$

3. $\varphi(x) = -\frac{\pi^2\lambda}{8(\lambda-1)} + \frac{1}{\sqrt{1-x^2}}, \lambda \neq 1.$
4. $\varphi(x) = \frac{1}{1-\lambda\Gamma(p+1)}.$
5. $\varphi(x) = \frac{2\lambda^2 x + \left(\frac{\lambda^2}{4}+\lambda\right)\ln}{1+\frac{29}{48}\lambda^2} + \frac{6}{5}(1-4x).$
6. $\varphi(x) = \frac{2}{2-\lambda}\sin x,\ \lambda \neq 2.$
7. $\varphi(x) = \lambda\pi^3 \sin x + x.$
8. $\varphi(x) = 2\left(\frac{2\cos x + \pi\lambda \sin}{4+\pi^2\lambda^2}\right).$
9. $\varphi(x) = \pi\lambda \sin x + \cos x.$
10. $\varphi(x) = \frac{15}{32}(x+1)^2 + \frac{5}{16}.$

Exercise (11b)

1. $\varphi_1(x) \equiv 0,\ \varphi_2(x) = \frac{7}{2}x^2,\ \varphi_{3,4}(x) = \pm\frac{15}{4\sqrt{7}}x + \frac{5}{4}x^2.$
2. $\varphi_1(x) \equiv 0, \varphi_{2,3}(x) = \pm 3x^2.$
3. $\varphi(x) \equiv 0.$

11.8 SUGGESTED READINGS:

1. K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering*, Cambridge University Press, 2006 (Third Edition). ISBN-978-0521679718.
2. F. G. Tricomi, *Integral Equations*, Dover Publications, 1985. ISBN-978-0486648286.
3. Rainer Kress, *Linear Integral Equations*, Springer, 2014 (Third Edition). ISBN-978-1447171474.
4. I. G. Petrovsky, *Lectures on Partial Differential Equations*, Dover Publications, 2012. ISBN-978-0486659640.
5. M. A. Krasnoselskii, *Integral Equations of the First Kind: Invariant Imbedding Method and Applications*, CRC Press, 1994. ISBN-978-2884490651.

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LESSON- 12

CHARACTERISTIC NUMBERS AND EIGENFUNCTIONS OF FREDHOLM INTEGRAL EQUATIONS

OBJECTIVE:

- To understand the concept of characteristic numbers and eigenfunctions for Fredholm integral equations.
- To explore both degenerate and general kernel cases in finding eigenfunctions and corresponding eigenvalues.
- To study properties of symmetric and skew-symmetric kernels and their implications on solutions.
- To examine bifurcation points in nonlinear integral equations and their significance in applications like stability analysis.

STRUCTURE:

12.1 Introduction

12.2 Fundamentals of Characteristic Numbers and Eigenfunctions

12.3 Fredholm Integral Equations with Difference Kernels

12.4 Extremal Properties of Characteristic Numbers and Eigenfunctions

12.5 Bifurcation Points

12.6 Summary:

12.7 Technical Terms:

12.8 Self-Assessment questions

12.9 Suggested Readings

12.1 INTRODUCTION:

In solving integral equations, the analysis of characteristic numbers and eigenfunctions provides key insights into the solution structure. Such spectral properties inform us about the solvability of homogeneous equations and the stability of physical systems modelled by such equations. This lesson introduces methods for deriving these values and explores advanced topics like bifurcation theory and kernel symmetry.

12.2 FUNDAMENTALS OF CHARACTERISTIC NUMBERS AND EIGENFUNCTIONS:

We introduce the fundamental definitions of characteristic numbers, eigenfunctions, and the index associated with integral equations. The characteristic numbers and their corresponding eigenfunctions are then determined for specific types of integral equations.

Consider the homogeneous Fredholm integral equation of the second kind

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = 0 \tag{12.1}$$

always has the obvious solution $\varphi(x) \equiv 0$, which is called the zero (trivial) solution.

12.2.1 Definition:

The values of the parameter λ for which the integral equation (12.1) admits nontrivial solutions $\varphi(x) \not\equiv 0$ are called the *characteristic numbers* of the equation or of the kernel $K(x,t)$. Any such nonzero solution $\varphi(x) \not\equiv 0$ corresponding to a characteristic number λ is referred to as an *eigenfunction.*

- If the kernel $K(x,t)$ is continuous in the square $\Omega\{a \le x, t \le b\}$, or is quadratically summable in Ω, and the numbers a and b are finite, then for every characteristic number λ , there exists only a finite set of linearly independent eigenfunctions; the number of such functions is called the *index* of the characteristic number. Different characteristic numbers can have different indices.

12.2.2 Note:

- For an equation with degenerated kernel

$$\varphi(x) - \lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x)b_k(t)\right] \varphi(t)dt = 0 \tag{12.2}$$

the characteristic numbers are roots of the algebraic equation

$$\Delta(\lambda) = \begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots -\lambda a_{2n} \\ \dots & \dots & \dots \quad \dots \\ -\lambda a_{n1} & -\lambda a_{n2} \quad \dots & 1-\lambda a_{nn} \end{vmatrix} = 0 \tag{12.3}$$

the degree of which is $p \le n$. Here, $\Delta(\lambda)$ is the determinant of the linear homogeneous system

$$\left.\begin{aligned} (1-\lambda a_{11})C_1 - \lambda a_{12}C_2 - \dots - \lambda a_{1n}C_n &= 0 \\ -\lambda a_{21}C_1 + (1-\lambda a_{22})C_2 - \dots - \lambda a_{2n}C_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ -\lambda a_{n1}C_1 - \lambda a_{n2}C_2 - \dots + (1-\lambda a_{nn})C_n &= 0 \end{aligned}\right\} \tag{12.4}$$

where the quantities a_{mk} and C_m $(k, m = 1, 2, \cdots, n)$ have the same meaning as in the preceding section.

- If equation (12.3) has p roots $1 \le p \le n$, then the integral equation (12.2) has p characteristic numbers; to each characteristic number λ_m $(m = 1, 2, \dots, p)$ there corresponds a nonzero solution

$$\begin{array}{llll} C_1^{(1)}, & C_2^{(1)}, & \dots & C_n^{(1)} \to \lambda_1, \\ C_1^{(2)}, & C_2^{(2)} & \dots & C_n^{(2)} \to \lambda_2, \\ \dots & \dots & \dots & \dots \\ C_1^{(p)}, & C_2^{(p)}, & \dots & C_n^{(p)} \to \lambda_p. \end{array}$$

of the system (12.4). The nonzero solutions of the integral equation (12.2) corresponding to these solutions, i.e., the eigenfunctions, will be of the form

$$\varphi_1(x) = \sum_{k=1}^{n} C_k^{(1)} a_k(x),$$

$$\varphi_2(x) = \sum_{k=1}^{n} C_k^{(2)} a_k(x), \cdots, \varphi_p(x) = \sum_{k=1}^{n} C_k^{(p)} a_k(x)$$

- An integral equation with degenerate kernel has at most *n* characteristic numbers and (corresponding to them) eigenfunctions.
- In the case of an arbitrary (nondegenerate) kernel, the characteristic numbers are zeros of the Fredholm determinant $D(\lambda)$, i.e., are poles of the resolvent kernel $R(x,t;\lambda)$. It then follows, in particular, that the Volterra integral equation

$$\varphi(x) - \lambda \int_0^x K(x,t)\varphi(t)dt = 0$$

where $K(x,t) \in L_2(\Omega_0)$ has no characteristic numbers.

- Eigenfunctions are determined to within a multiplicative constant; that is, if $\varphi(x)$ is an eigenfunction corresponding to some characteristic number λ, then $C\varphi(x)$, where C is an arbitrary constant, is also an eigenfunction which corresponds to the same characteristic number λ.

12.2.3 Example: Find the characteristic numbers and eigenfunctions of the integral equation

$$\varphi(x) - \lambda \int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t)\,\varphi(t)dt = 0.$$

Solution. We have

$$\varphi(x) = \lambda \cos^2 x \int_0^\pi \varphi(t) \cos 2t\, dt + \lambda \cos 3x \int_0^\pi \varphi(t) \cos^3 t dt$$

Introducing the notations

$$C_1 = \int_0^\pi \varphi(t) \cos 2t\, dt, \qquad C_2 = \int_0^\pi \varphi(t) \cos^3 t dt \tag{12.5}$$

we get

$$\varphi(x) = C_1 \lambda \cos^2 x + C_2 \lambda \cos 3x \tag{12.6}$$

Substituting (12.6) into (12.5), we obtain a linear system of homogeneous equations:

$$\left.\begin{aligned} C_1\left(1-\lambda\int_0^{\pi} cos^2t\,cos\,2t\,dt\right)-C_2\lambda\int_0^{\pi}\cos 3t\,cos\,2t\,dt=0 \\ -C_1\lambda\int_0^{\pi} cos^5t\,dt+C_2\left(1-\lambda\int_0^{\pi} cos^3t\,cos\,3t\,dt\right)=0 \end{aligned}\right\} \qquad (12.7)$$

But since

$$\int_0^{\pi} cos^2t\,cos\,2t\,dt=\frac{\pi}{4},\quad \int_0^{\pi}\cos 3t\,cos\,2t\,dt=0,$$

$$\int_0^{\pi} cos^5\,t\,dt=0\,,\int_0^{\pi} cos^3t\,cos\,3t\,dt=\frac{\pi}{8}$$

it follows that system (12.7) takes the form

$$\left.\begin{aligned}\left(1-\tfrac{\lambda\pi}{4}\right)C_1=0,\\ \left(1-\tfrac{\lambda\pi}{8}\right)C_2=0\end{aligned}\right\} \qquad (12.8)$$

The equation for finding characteristic numbers will be

$$\begin{vmatrix} 1-\dfrac{\lambda\pi}{4} & 0 \\ 0 & 1-\dfrac{\lambda\pi}{8}\end{vmatrix}=0$$

The characteristic numbers are $\lambda_1=\frac{4}{\pi},\ \lambda_2=\frac{8}{\pi}$.

For $\lambda=\frac{4}{\pi}$, system (12.8) becomes

$$\begin{cases}0.\,C_1=0,\\ \dfrac{1}{2}.\,C_2=0\end{cases}$$

whence $C_2=0, C_1$ is arbitrary. The eigenfunction will be $\varphi_1(x)=C_1\lambda cos^2x$ or setting $C_1\lambda=1$ we get $\varphi_1(x)=cos^2x$.

For $\lambda=\frac{8}{\pi}$, system (12.4) is of the form

$$\begin{cases}(-1).\,C_1=0,\\ \quad 0.\,C_2=0\end{cases}$$

Whence $C_1=0, C_2$ is arbitrary and, hence, the eigenfunction will be $\varphi_2(x)=C_2\lambda\,cos\,3x$, or assuming $C_2\lambda=1$, we get $\varphi_2(x)=cos\,3x$.

Thus, the characteristic numbers are

$$\lambda_1=\frac{4}{\pi},\qquad \lambda_2=\frac{8}{\pi}.$$

and the corresponding eigenfunctions are

$$\varphi_1(x)=cos^2x,\qquad \varphi_2(x)=cos\,3x.$$

12.2.4 Note: A homogeneous Fredholm integral equation may, generally, have no characteristic numbers and eigenfunctions, or it may not have any real characteristic numbers and eigenfunctions.

12.2.5 Example: The homogeneous integral equation

$$\varphi(x) - \lambda \int_0^1 (3x-2)\, t\, \varphi(t)dt = 0$$

has no characteristic numbers and eigenfunctions. Indeed, we have

$$\varphi(x) = \lambda(3x-2) \int_0^1 t\, \varphi(t)dt$$

Putting

$$C = \int_0^1 t\, \varphi(t)dt \tag{12.9}$$

we get

$$\varphi(x) = C\lambda(3x-2) \tag{12.10}$$

Substituting (12.10) into (12.9), we get

$$\left[1 - \lambda \int_0^1 (3t^2 - 2t)\, dt\right] . C = 0 \tag{12.11}$$

equation (12.11) yields $C = 0$ and, hence, $\varphi(x) \equiv 0$.

And so, for any λ, this homogeneous equation has only one zero solution $\varphi(x) \equiv 0$ and, hence, it does not have any characteristic numbers or eigenfunctions.

12.2.6 Example: The equation

$$\varphi(x) - \lambda \int_0^1 \left(\sqrt{x}t - \sqrt{t}x\right) \varphi(t)dt = 0$$

does not have real characteristic numbers and eigenfunctions.

We have

$$\varphi(x) = C_1\lambda\sqrt{x} - C_2\lambda x \tag{12.12}$$

where

$$C_1 = \int_0^1 t\, \varphi(t)dt\,, \qquad C_2 = \int_0^1 \sqrt{t}\, \varphi(t)dt \tag{12.13}$$

Substituting (12.12) into (12.13), we get (after some simple manipulations) the system· of algebraic equations

$$\left.\begin{aligned}\left(1-\frac{2\lambda}{5}\right)C_1+\frac{\lambda}{3}C_2=0,\\ -\frac{\lambda}{2}C_1+\left(1+\frac{2\lambda}{5}\right)C_2=0\end{aligned}\right\} \qquad (12.14)$$

The determinant of this system is

$$\Delta(\lambda)=\begin{vmatrix}1-\frac{2\lambda}{5} & \frac{\lambda}{3}\\ -\frac{\lambda}{2} & 1+\frac{2\lambda}{5}\end{vmatrix}=1+\frac{\lambda^2}{150}$$

For real λ, it does not vanish, so that from (12.14) we get $C_1=0$ and $C_2=0$ and, hence, for all real λ the equation has only one solution, namely, the zero solution $\varphi(x)\equiv 0$. Thus, equation (12.12) does not have real characteristic numbers or eigenfunctions.

12.2.7 Example: Find the characteristic numbers and eigenfunctions for the homogeneous integral equations with degenerate kernels

$$\varphi(x)-\lambda\int_0^{\frac{\pi}{4}}\sin^2 x\varphi(t)dt=0 \qquad (12.15)$$

Solution. We can rewrite the equation (12.15) as

$$\varphi(x)=\lambda\sin^2 x\, C \qquad (12.16)$$

where

$$C=\int_0^{\frac{\pi}{4}}\varphi(t)dt \qquad (12.17)$$

Substituting (12.16) into (12.17), we get

$$C=\lambda C\int_0^{\frac{\pi}{4}}\sin^2 t\, dt$$

Implies

$$C=\lambda C\frac{\pi-2}{8}$$

for a nontrivial solution ($C\neq 0$), we get

$$\lambda=\frac{8}{\pi-2}$$

and corresponding eigenfunction is $\varphi(x)=\sin^2 x$.

12.2.8 Note: If the n^{th} iterated kernel $K_n(x,t)$ of the kernel $K(x,t)$ is symmetric, then it may be asserted that $K(x,t)$ has at least one characteristic number (real or complex) and that the n^{th} degrees of all characteristic numbers are real numbers. In particular, for the skew-symmetric kernel $K(x,t)=-K(t,x)$ all characteristic numbers are pure imaginary $\lambda=\beta i$, where β is real

12.2.9 Definition: The kernel $K(x,t)$ of the integral equation (12.1) is called *symmetric* if the condition $K(x,t)=K(t,x)$ $(a\leq x,t\leq b)$ is satisfied.

12.2.10 Theorems: The following theorems hold for the Fredholm integral equation (12.1) with symmetric kernel $K(x,t)$:

12.2.10.1 Theorem 1. Equation (12.1) has at least one real characteristic number.

12.2.10.2 Theorem 2. To every characteristic number λ there corresponds a finite number q of linearly independent eigenfunctions of equation (12.1), and

$$Sup\, q \leq \lambda^2 B^2$$

where,

$$B^2 = \int_a^b \int_a^b K^2(x,t)dxdt$$

The number q is called the *index* or *multiplicity* of the characteristic number.

12.2.10.3 Theorem 3. Every pair of eigenfunctions $\varphi_1(x)$, $\varphi_2(x)$ corresponding to different characteristic numbers, $\lambda_1 \neq \lambda_2$, is orthogonal; i.e.,

$$\int_a^b \varphi_1(x)\varphi_2(x)dx = 0$$

12.2.10.4 Theorem: There is a finite number of characteristic numbers in every finite interval of the λ −axis. The upper bound for a number m of characteristic numbers lying in an interval $-l < \lambda < l$ is defined by the inequality

$$m \leq l^2 B^2$$

12.2.10.5 Note: When the kernel $K(x,t)$ of the integral equation (12.1) is the Green's function of some homogeneous Sturm-Liouville problem, finding the characteristic numbers and eigenfunctions reduces to the solution of the indicated Sturm-Liouville problem.

12.2.11 Example: Find the characteristic numbers and eigenfunctions of the homogeneous equation

$$\varphi(x) - \lambda \int_0^\pi K(x,t)\varphi(t)dt = 0$$

where,

$$K(x,t) = \begin{cases} \cos x \sin t\,, & if\ 0 \leq x \leq t, \\ \cos t \sin x\,, & if\ t \leq x \leq \pi \end{cases}$$

Solution. Represent the equation in the form

$$\varphi(x) = \lambda \int_0^x K(x,t)\varphi(t)dt + \lambda \int_x^\pi K(x,t)\varphi(t)dt$$

or

$$\varphi(x) = \lambda \sin x \int_0^x \varphi(t) \cos t \, dt + \lambda \cos x \int_x^\pi \varphi(t) \sin t \, dt \tag{12.18}$$

Differentiating both sides of (12.18), we get

$$\varphi'(x) = \lambda \cos x \int_0^x \varphi(t) \cos t \, dt + \lambda \sin x \cos x \, \varphi(x)$$
$$- \lambda \sin x \int_x^\pi \varphi(t) \sin t \, dt - \lambda \sin x \cos x \, \varphi(x)$$

or

$$\varphi'(x) = \lambda \cos x \int_0^x \varphi(t) \cos t \, dt - \lambda \sin x \int_x^\pi \varphi(t) \sin t \, dt \tag{12.19}$$

differentiating again, we get

$$\varphi''(x) = -\lambda \sin x \int_0^x \varphi(t) \cos t \, dt + \lambda \cos^2 x \, \varphi(x) - \lambda \cos x \int_x^\pi \varphi(t) \sin t \, dt$$
$$+ \lambda \sin^2 x \, \varphi(x)$$

$$\varphi''^{(x)} = \lambda \, \varphi(x) - \left[\lambda \sin x \int_0^x \varphi(t) \cos t \, dt + \lambda \cos x \int_x^\pi \varphi(t) \sin t \, dt\right]$$

The expression in the square brackets is equal to $\varphi(x)$ so that

$$\varphi''(x) = \lambda\varphi(x) - \varphi(x)$$

From (12.18) and (12.19) we find that

$$\varphi(\pi) = 0, \qquad \varphi'(0) = 0$$

Thus, the given integral equation reduces to the following boundary-value problem:

$$\varphi''(x) - (\lambda - 1)\varphi(x) = 0 \tag{12.20}$$
$$\varphi(\pi) = 0, \qquad \varphi'(0) = 0 \tag{12.21}$$

The three following cases are possible:

1. $\lambda - 1 = 0$ *or* $\lambda = 1$. Equation (12.20) takes the form $\varphi''(x) = 0$. Its general solution will be $\varphi(x) = C_1 x + C_2$. Utilizing the boundary conditions (12.21), we obtain the system

$$\begin{cases} C_1\pi + C_2 = 0, \\ C_1 = 0 \end{cases}$$

   which has a unique solution: $C_1 = 0, C_2 = 0$, and hence the integral equation has only the trivial solution

$$\varphi(x) \equiv 0$$

2. $\lambda - 1 > 0$ or $\lambda > 1$. The general solution of equation (12.20) is of the form

$$\varphi(x) = C_1 cosh\sqrt{\lambda - 1}x + C_2 sinh\sqrt{\lambda - 1}x$$

whence

$$\varphi'(x) = \sqrt{\lambda-1}\left(C_1 sinh\sqrt{\lambda-1}x + C_2 cosh\sqrt{\lambda-1}x\right)$$

For finding the values of C_1 and C_2, the boundary conditions yield the system

$$\begin{cases} C_1\, cosh\,\pi\sqrt{\lambda-1} + C_2\, sinh\,\pi\sqrt{\lambda-1} = 0, \\ \qquad\qquad C_2 = 0 \end{cases}$$

The system has a unique solution: $C_1 = 0, C_2 = 0$ The integral equation has the trivial solution $\varphi(x) \equiv 0$. Thus, for $\lambda \geq 1$ the integral equation has no characteristic numbers and, hence, no eigenfunctions.

3. $\lambda - 1 < 0$ or $\lambda < 1$. The general solution of equation (12.20) is of the form

$$\varphi(x) = C_1 cos\sqrt{1-\lambda}x + C_2 sin\sqrt{1-\lambda}x$$

whence

$$\varphi'(x) = \sqrt{1-\lambda}\left(-C_1 sin\sqrt{1-\lambda}x + C_2 cos\sqrt{1-\lambda}x\right)$$

In this case, for finding C_1 and C_2 the boundary conditions (12.21) yield the system

$$\left.\begin{aligned} C_1\, cos\,\pi\sqrt{1-\lambda} + C_2\, sin\,\pi\sqrt{1-\lambda} = 0, \\ \sqrt{1-\lambda}C_2 = 0 \end{aligned}\right\} \qquad (12.22)$$

The determinant of this system is

$$\Delta\lambda = \begin{vmatrix} cos\,\pi\sqrt{1-\lambda} & sin\,\pi\sqrt{1-\lambda} \\ 0 & \sqrt{1-\lambda} \end{vmatrix}$$

Setting it equal to zero, we get an equation for finding the characteristic numbers:

$$\begin{vmatrix} cos\,\pi\sqrt{1-\lambda} & sin\,\pi\sqrt{1-\lambda} \\ 0 & \sqrt{1-\lambda} \end{vmatrix} = 0 \qquad (12.23)$$

or $\sqrt{1-\lambda}\ cos\,\pi\sqrt{1-\lambda} = 0$. By assumption $\sqrt{1-\lambda} \neq 0$ and so $cos\,\pi\sqrt{1-\lambda} = 0$. Whence we find that $\pi\sqrt{1-\lambda} = \frac{\pi}{2} + \pi n$, where *n* is any integer. All the roots of equation (12.23) are given by the formula

$$\lambda_n = 1 - \left(n + \frac{1}{2}\right)^2$$

For values $\lambda = \lambda_n$ the system (12.22) takes the form

$$\begin{cases} C_1 . 0 = 0, \\ C_2 = 0 \end{cases}$$

It has an infinite number of nonzero solutions

$$\begin{cases} C_1 = C, \\ C_2 = 0 \end{cases}$$

where C is an arbitrary constant. Hence, the original integral equation also has an infinity of solutions of the form

$$\varphi(x) = C\, cos\left(n + \frac{1}{2}\right)x$$

which are eigenfunctions of this equation. Hence, the characteristic numbers and eigenfunctions of the given integral equation will be

$$\lambda_n = 1 - \left(n + \frac{1}{2}\right)^2, \quad \varphi_n(x) = cos\left(n + \frac{1}{2}\right)x$$

where n is any integer.

12.2.12 Example: Find the characteristic numbers and eigenfunctions of the homogeneous equation

$$\varphi(x) - \lambda \int_0^1 K(x,t)\varphi(t)dt = 0$$

where,

$$K(x,t) = \begin{cases} x(t-1), & 0 \le x \le t, \\ t(x-1), & t \le x \le 1. \end{cases}$$

Solution. Represent the equation in the form

$$\varphi(x) = \lambda\left[\int_0^x t(x-1)\varphi(t)dt + \int_x^1 x(t-1)\varphi(t)dt\right]$$

Differentiating on both sides we get

$$\varphi'(x) = \lambda \int_0^x t\, \varphi(t)dt + \lambda \int_x^1 (t-1)\varphi(t)dt$$

Differentiating again, we get

$$\varphi''^{(x)} = \lambda x\varphi(x) - \lambda(x-1)\varphi(x)$$

$$\varphi''(x) - \lambda\varphi(x) = 0$$

Thus, the given integral equation reduces to the following boundary-value problem:

$$\varphi''(x) - \lambda\varphi(x) = 0$$

$$\varphi(0) = 0, \qquad \varphi(1) = 0$$

The three following cases are possible:

1. If $\lambda = 0,$ then $\varphi''(x) = 0.$
   Its general solution will be $\varphi(x) = C_1 + C_2 x.$ Utilizing the boundary conditions we get $C_1 = 0, C_2 = 0,$ which is trivial $(\varphi(x) \equiv 0)$
2. If $\lambda = k^2,$ The general solution is of the form

$$\varphi(x) = C_1 e^{kx} + C_2 e^{-k}$$

Which is also trivial solution.

3. If $\lambda = -k^2$, the general solution is of the form

$$\varphi(x) = C_1 \cos kx + C_2 \sin kx$$

Hence by using boundary conditions we get,

$$\begin{cases} C_1 = 0, \\ C_2 \neq 0 \end{cases}$$

It has an infinite number of nonzero solutions

$$\begin{cases} C_1 = 0, \\ C_2 = C \end{cases}$$

where C is an arbitrary constant. Hence, the original integral equation also has an infinity of solutions of the form

$$\varphi(x) = C \sin (n\pi)\, x$$

which are eigenfunctions of this equation. Hence, the characteristic numbers and eigenfunctions of the given integral equation will be

$$\lambda_n = -n^2\pi^2, \quad \varphi_n(x) = \sin (n\pi)\, x$$

where n is any integer.

12.3 FREDHOLM INTEGRAL EQUATIONS WITH DIFFERENCE KERNELS:

We define difference kernels and show that their characteristic numbers are the Fourier coefficients of even kernels, with suitable examples.

Suppose we have the integral equation

$$\varphi(x) = \lambda \int_{-\pi}^{\pi} K(x,t)\varphi(t)dt \tag{12.24}$$

where the kernel $K(x)$ $(-\pi \leq x \leq \pi)$ is an even function which is periodically extended to the entire x-axis so that

$$K(x - t) = K(t - x) \tag{12.25}$$

It can be shown that the eigenfunctions of equation (12.24) are

$$\left.\begin{aligned} \varphi_n^{(1)}(x) &= \cos nx \quad (n = 1, 2, \ldots), \\ \varphi_n^{(2)}(x) &= \sin nx \quad (n = 1, 2, \ldots). \end{aligned}\right\} \tag{12.26}$$

and the corresponding characteristic numbers are

$$\lambda_n = \frac{1}{\pi a_n} \qquad (n = 1, 2, \ldots) \tag{12.27}$$

where a_n are the Fourier coefficients of the function $K(x)$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} K(x)\cos nx\, dx \quad (n = 1, 2, \dots) \tag{12.28}$$

Thus, to every value of λ_n there correspond two linearly independent eigenfunctions $\cos nx$, $\sin nx$ so that each λ_n is a double characteristic number. The function $\varphi_0(x) \equiv 1$ is also an eigenfunction of equation (12.24) corresponding to the characteristic number

$$\lambda_0 = \frac{1}{\pi a_0}, \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} K(x)dx$$

We shall now show that, for example, $\cos nx$ is an eigenfunction of the integral equation

$$\varphi(x) = \frac{\pi^{-1}}{a_n}\int_{-\pi}^{\pi} K(x-t)\varphi(t)dt \tag{12.29}$$

Where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} K(x)\cos nx\, dx$$

Making the substitution $x - t = z$, we find

$$\int_{-\pi}^{\pi} K(x-t)\cos nt\, dt = -\int_{x+\pi}^{x-\pi} K(z)\cos n(x-z)\, dz$$

$$= \cos nx \int_{x-\pi}^{x+\pi} K(z)\cos nz\, dz + \sin nx \int_{x-\pi}^{x+\pi} K(z)\sin nz\, dz$$

$$= \pi a_n \cos nx$$

since by virtue of the evenness of $K(x)$ the second integral is zero, and the first integral is a Fourier coefficient a_n multiplied by π in the expansion of the even function $K(x)$.

Thus,

$$\cos nx = \frac{1}{\pi a_n}\int_{-\pi}^{\pi} K(x-t)\cos nt\, dt$$

and this means that $\cos nx$ is an eigenfunction of equation (12.29).

Similarly, we establish the fact that $\sin nx$ is an eigenfunction of the integral equation (12.29) corresponding to the same characteristic number $\frac{1}{\pi a_n}$.

12.3.1 Find the eigenfunction and the corresponding characteristic numbers of the equation

$$\varphi(x) = \lambda \int_{-\pi}^{\pi} \cos^2(x-t)\,\varphi(t)$$

Solution.

where the kernel $K(x,t) = \cos^2(x-t)$ is an even function i. e.,

$$\cos^2(x-t) = \cos^2(t-x)$$

Implies $K(x,t) = \frac{1}{2} + \frac{1}{2}\cos(2(x-t))$. The function $\varphi_0(x) \equiv 1$ is also an eigenfunction of the given integral equation corresponding to the characteristic number $\lambda_0 = \frac{1}{\pi}$,

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} K(x)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx = 1$$

and the corresponding characteristic numbers are

$$\lambda_2 = \frac{1}{2\pi} \qquad (n = 1, 2, \dots)$$

where a_n are the Fourier coefficients of the function $K(x)$:

$$a_1 = a_3 = a_5 = \cdots = 0,$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\cos 2x\, dx = \frac{1}{2}$$

It can be shown that the eigenfunctions of given integral equation are

$$\varphi_0(x) \equiv 1;\ \varphi_1^{(1)}(x) = \cos 2x,\ \varphi_1^{(2)}(x) = \sin 2x.$$

12.4 EXTREMAL PROPERTIES OF CHARACTERISTIC NUMBERS AND EIGENFUNCTIONS:

We define the concept of the maximum value of a double integral equation and provide examples to demonstrate the process of determining this maximum

Consider the double integral (Hilbert's integral) equation

$$|K\varphi, \varphi| = \left| \int_a^b \int_a^b K(x,t)\varphi(x)\varphi(t)dxdt \right| \tag{12.30}$$

where $K(x,t) = K(t,x)$ is a symmetric kernel of some integral equation, on the set of normalized functions $\varphi(x)$, i.e.,

$$(\varphi, \varphi) = \int_a^b \varphi^2(x)dx = 1$$

has a maximum equal to

$$max|K\varphi,\varphi| = \frac{1}{|\lambda_1|} \tag{12.31}$$

where λ_1 is the least (in absolute value) characteristic number of the kernel $K(x,t)$. The maximum is attained for $\varphi(x) = \varphi_1(x)$, which is the eigenfunction of the kernel corresponding to λ_1.

12.4.1 Example: Find the maximum of

$$|K\varphi,\varphi| = \left|\int_0^\pi\int_0^\pi K(x,t)\varphi(x)\varphi(t)dxdt\right|$$

provided

$$(\varphi,\varphi) = \int_0^\pi \varphi^2(x)dx = 1$$

If

$$K(x,t) = \cos x\cos 2t + \cos t\cos 2x + 1.$$

Solution. Solving the homogeneous integral equation

$$\varphi(x) = \lambda\int_0^\pi (\cos x\cos 2t + \cos t\cos 2x + 1)\varphi(t)dt$$

as an equation with a degenerate kernel, we find the characteristic umbers $\lambda_1 = \frac{1}{\pi}$ and $\lambda_2, \lambda_3 = \pm\frac{2}{\pi}$ and the corresponding eigenfunctions $\varphi_1(x) = C_1, \varphi_2(x) = C_2(\cos x + \cos 2x)$, $\varphi_3(x) = C_3(\cos x - \cos 2x)$, where C_1, C_2 and C_3 are arbitrary constants.

The smallest (in absolute value) characteristic number is $\lambda_1 = \frac{1}{\pi}$, to which corresponds the eigenfunction $\varphi_1(x) = C_1$. From the normalization condition $(\varphi,\varphi) = 1$, we find $C_1 = \pm\frac{1}{\sqrt{2\pi}}$. Hence

$$max\left|\int_0^\pi\int_0^\pi (\cos x\cos 2t + \cos t\cos 2x + 1)\varphi(t)dt\right| = 2\pi$$

and it is attained on the functions $\varphi(x) = \pm\frac{1}{\sqrt{2\pi}}$.

12.4.2 Find the maximum of

$$|K\varphi,\varphi| = \left|\int_0^1\int_0^1 K(x,t)\varphi(x)\varphi(t)dxdt\right|$$

provided

$$(\varphi,\varphi) = \int_0^1 \varphi^2(x)dx = 1$$

if

$$K(x,t) = xt.$$

Solution. Solving the homogeneous integral equation

$$\varphi(x) = \lambda \int_0^1 (xt)\varphi(t)dt$$

It can be rewritten as

$$\varphi(x) = \lambda x C$$

Where

$$C = \int_0^1 t\,\varphi(t)dt$$

as an equation with a degenerate kernel, we find the characteristic umbers $\lambda = \frac{1}{3}$ and the corresponding eigenfunctions $\varphi(x) = \frac{x}{3}C$.

The characteristic number is $\lambda = \frac{1}{3}$, to which corresponds the eigenfunction $\varphi(x) = C$. From the normalization condition $(\varphi, \varphi) = 1$, we find $C = \pm 3\sqrt{3}$. Hence

$$max\left|\int_0^1 \int_0^1 (xt)\varphi(t)dt\right| = \frac{3}{4}$$

and it is attained on the functions $\varphi(x) = \pm\sqrt{3}x$.

12.5 BIFURCATION POINTS:

Suppose we have a nonlinear integral equation

$$\varphi(x) = \lambda \int_a^b K(x,t,\varphi(t))dt \qquad (12.32)$$

Let $\varphi(x) \equiv 0$ be a solution of the equation, and

$$K(x,t,0) \equiv 0$$

By analogy with linear integral equations, the nonzero solutions $\varphi(x) \equiv 0$ of equation (12.32) are called *eigenfunctions* and the corresponding values of the parameter λ are called *characteristic numbers* of the equation.

12.5.1 Definition: The number λ_0 is called a *bifurcation point* of the nonlinear equation (12.32) if for any $\varepsilon > 0$ there is a characteristic number λ of equation (12.32) such that $|\lambda - \lambda_0| < \varepsilon$, and to this characteristic number there corresponds at least one eigenfunction $\varphi(x)$ $(\varphi(x) \not\equiv 0)$ with norm less than ε: $\|\varphi\| < \varepsilon$.

12.5.2 Note: In problems of technology and physics involving conditions of stability, bifurcation points determine critical forces. Thus, the problem of the. bending of a rectilinear rod of unit length and variable rigidity $\rho(x)$ under the action of a force P leads to the solution of the following nonlinear integral equation:

$$\varphi(x) = P\rho(x)\int_0^1 K(x,t)\varphi(t)\sqrt{1-\left[\int_0^1 K'_x(x,t)\varphi(t)dt\right]^2}\,dt \tag{12.32}$$

where $\varphi(x)$ is the unknown function. For small P, equation (12.32) has a unique zero solution in the space $C\,[0,1]$. This means that for small P the rod does not bend. However, a deflection occurs for forces greater than the so-called critical force of Euler. Euler's critical force is the bifurcation value.

12.5.3 Example: To illustrate how to find bifurcation points, let us consider the following nonlinear equation

$$\varphi(x) = \lambda\int_0^1 [\varphi(t) + \varphi^3(t)]dt \tag{12.33}$$

Put

$$C = \int_0^1 [\varphi(t) + \varphi^3(t)]dt$$

Then

$$\varphi(x) = C\lambda \tag{12.34}$$

and equation (12.33) reduces to the algebraic equation

$$C = \lambda C + \lambda^3 C^3 \tag{12.35}$$

From (12.35) we get

$$C_1 = 0, C_{2,3} = \pm\sqrt{\frac{1-\lambda}{\lambda^3}}$$

whence, by (12.34),

$$\varphi_1 \equiv 0,\ \ \varphi_{2,3} = \pm\sqrt{\frac{1-\lambda}{\lambda}}$$

Thus, for any $0 < \lambda < 1$, equation (12.33) admits real nonzero solutions. For $\lambda = 1$ it has only the zero solution $\varphi \equiv 0$

Thus, for any $0 < \varepsilon < 1$, the number $\lambda = 1 - \varepsilon$ is a characteristic number of equation (12.33) to which there correspond two eigenfunctions:

$$\varphi_1 = \frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon}}; \qquad \varphi_2 = -\frac{\sqrt{\varepsilon}}{\sqrt{1-\varepsilon}}$$

where $\varepsilon = 1 - \lambda$ Hence, the point $\lambda_0 = 1$ is a bifurcation point of equation (12.33). One can also speak of bifurcation points of nonzero solutions of nonlinear integral equations.

12.5.4 Example: Find bifurcation points of the zero solution of the integral equation

$$\varphi(x) = \lambda\int_0^1 xt\,[\varphi(t) + \varphi^3(t)]dt \tag{12.36}$$

Put

$$C = \int_0^1 t\,[\varphi(t) + \varphi^3(t)]dt$$

Then

$$\varphi(x) = \lambda x C \tag{12.37}$$

and equation (12.36) reduces to the algebraic equation

$$C = \frac{5\lambda C + 3\lambda^3 C^3}{15} \tag{12.38}$$

From (12.38) we get

$$C_1 = 0, C_{2,3} = \pm\sqrt{\frac{15 - 5\lambda}{3\lambda^3}}$$

whence, by (12.37),

$$\varphi_1 \equiv 0,\ \ \varphi_{2,3} = \pm\sqrt{\frac{15 - 5\lambda}{3\lambda}}$$

Thus, for any $0 < \lambda < 3$, equation (12.36) admits real nonzero solutions. For $\lambda = 3$ it has only the zero solution $\varphi \equiv 0$

Thus, for any $0 < \varepsilon < 3$, the number $\lambda = 3 - \varepsilon$ is a characteristic number of equation (12.36) to which there correspond two eigenfunctions:

$$\varphi_1 = \frac{\sqrt{5\varepsilon}}{\sqrt{3(3-\varepsilon)}}; \qquad \varphi_2 = -\frac{\sqrt{5\varepsilon}}{\sqrt{3(3-\varepsilon)}}$$

where $\varepsilon = 3 - \lambda$ Hence, the point $\lambda_0 = 3$ is a bifurcation point of equation (12.36). One can also speak of bifurcation points of nonzero solutions of nonlinear integral equations.

12.6 SUMMARY:

This lesson delves into the spectral theory of Fredholm integral equations, focusing on the concepts of characteristic numbers and eigenfunctions. It covers both degenerate and non-degenerate kernels, showing how these lead to systems of algebraic equations. It also presents symmetric and skew-symmetric kernels and explores bifurcation theory, where small changes in parameters cause the emergence of new solutions. Theoretical results and concrete examples highlight the link between integral equations and physical phenomena like stability.

12.7 TECHNICAL TERMS:

- **Characteristic Number:** Also called eigenvalue; a value of λ for which the Fredholm equation has non-trivial solutions.
- **Eigenfunction:** A non-zero solution corresponding to a characteristic number.

- **Degenerate Kernel:** A kernel that is a finite sum of separable functions, simplifying the integral equation.
- **Symmetric Kernel:** A kernel $K(x,t) = K(t,x)$; such kernels have real characteristic numbers.
- **Bifurcation Point:** A critical value of a parameter where the number or type of solutions to an equation changes.

12.8 SELF-ASSESSMENT QUESTIONS:

Exercise (12a): Find the characteristic numbers and eigenfunctions for the following homogeneous integral equations with degenerate kernels:

1. $$\varphi(x) - \lambda \int_0^{2\pi} \sin x \cos t \ \varphi(t)dt = 0.$$
2. $$\varphi(x) - \lambda \int_0^{2\pi} \sin x \sin t \ \varphi(t)dt = 0.$$
3. $$\varphi(x) - \lambda \int_0^{\pi} \cos(x+t) \ \varphi(t)dt = 0.$$
4. $$\varphi(x) - \lambda \int_0^{1} (45x^2 \ln t - 9t^2 \ln x) \ \varphi(t)dt = 0.$$
5. $$\varphi(x) - \lambda \int_0^{1} (2xt - 4x^2) \ \varphi(t)dt = 0.$$
6. $$\varphi(x) - \lambda \int_{-1}^{1} (5xt^3 + 4x^2t) \ \varphi(t)dt = 0.$$
7. $$\varphi(x) - \lambda \int_{-1}^{1} (5xt^3 + 4x^2t + 3xt) \ \varphi(t)dt = 0.$$
8. $$\varphi(x) - \lambda \int_{-1}^{1} (x \cosh t - t \sinh x) \ \varphi(t)dt = 0.$$
9.

$$\varphi(x) - \lambda \int_{-1}^{1} (x \cosh t - t^2 \sinh x)\, \varphi(t) dt = 0.$$

10.
$$\varphi(x) - \lambda \int_{-1}^{1} (x \cosh t - t \cosh x)\, \varphi(t) dt = 0.$$

Exercise (12b): Find the characteristic numbers and eigenfunctions of the homogeneous integral equations if their kernels are of the following form:

1. $K(x,t) = \begin{cases} t(x+1), & 0 \le x \le t, \\ x(t+1), & t \le x \le 1. \end{cases}$
2. $K(x,t) = \begin{cases} (x+1)(t-2), & 0 \le x \le t, \\ (t+1)(x-2), & t \le x \le 1. \end{cases}$
3. $K(x,t) = \begin{cases} \sin x \cos t\,, & 0 \le x \le t, \\ \sin t \cos x\,, & t \le x \le \frac{\pi}{2}. \end{cases}$
4. $K(x,t) = \begin{cases} \sin x \cos t\,, & 0 \le x \le t, \\ \sin t \cos x\,, & t \le x \le \pi. \end{cases}$
5. $K(x,t) = \begin{cases} \sin x \sin (t-1)\,, & -\pi \le x \le t, \\ \sin t \sin (x-1)\,, & t \le x \le \pi. \end{cases}$
6. $K(x,t) = \begin{cases} \sin\left(x + \frac{\pi}{4}\right) \sin\left(t - \frac{\pi}{4}\right), & 0 \le x \le t, \\ \sin\left(t + \frac{\pi}{4}\right) \sin\left(x - \frac{\pi}{4}\right), & t \le x \le \pi. \end{cases}$
7. $K(x,t) = e^{-|x-t|},\ 0 \le x \le 1,\ \ 0 \le t \le 1.$
8. $K(x,t) = \begin{cases} -e^{-t} \sinh x\,, & 0 \le x \le t, \\ -e^{-x} \sinh t\,, & t \le x \le 1. \end{cases}$
9. Show that if $\lambda_1, \lambda_2, \lambda_1 \ne \lambda_2$ are characteristic numbers of the kernel $K(x,t)$, then the eigenfunctions of the equations
$$\varphi(x) - \lambda_1 \int_a^b K(x,t)\varphi(t) dt = 0$$
$$\psi(x) - \lambda_2 \int_a^b K(x,t)\varphi(t) dt = 0$$
are orthogonal, i.e.,
$$\int_a^b \varphi(x)\psi(x) dt = 0$$
10. Show that if $K(x,t)$ is a symmetric kernel, then the second iterated kernel $K_2(x,t)$ has only positive characteristic numbers.
11. Show that if the kernel $K(x,t)$ is skew-symmetric, that is, $K(t,x) = -K(x,t)$ then all its characteristic numbers are pure imaginaries.
12. If the kernel $K(x,t)$ is symmetric, then

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^m} = A_m \qquad (m = 2, 3, \dots)$$

Where λ_n are characteristic numbers and A_m are the m^{th} traces of the kernel $K(x,t)$.

Taking advantage of the results of Problems 3 an 7 find the sums of the series:

a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b) $\sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^2}$

c) $\sum_{n=1}^{\infty} \frac{1}{(1+\mu_n^2)^2}$,

where μ_n are the roots of the equation $2\cot\mu = \mu - \frac{1}{\mu}$.

13. Find the eigenfunctions of the integral equations who resolvent kernels are defined by the following formulas:

a) $R(x,t;\, \lambda) = \frac{3-\lambda+3(1-\lambda)(2x-1)(2t-1)}{\lambda^2-4\lambda+3}$.

b) $R(x,t;\, \lambda) = \frac{(15-6\lambda)xt+(15-10\lambda)x^2t^2}{4\lambda^2-16\lambda+}$.

c) $R(x,t;\, \lambda) = \frac{4(5-2\lambda)[3-2\lambda+(3-6\lambda)xt]+5(4\lambda^2-8\lambda+3)(3x^2-1)(3t^2-1)}{4(1-2\lambda)(3-2\lambda)(5-2\lambda)}$

Exercise (12c):

1. Show that the symmetric kernel

$$K(x,t) = \frac{1}{2\pi}\frac{1-h^2}{1-2h\cos(x-t)+h^2} \qquad (-\pi \le x, t \le \pi)$$

has for $|h| < 1$ the eigenfunctions 1, $\sin nx, \cos nx$, which correspond to the characteristic numbers $1, \frac{1}{h^n}, \frac{1}{h^n}$.

2. Find the characteristic numbers and eigenfunctions of the integral equation

$$\varphi(x) = \lambda \int_{-\pi}^{\pi} K(x-t)\varphi(t)dt$$

where $K(x) = x^2(-\pi \le x \le \pi)$ is a periodic function with period 2π.

Exercise (12d):

1. Find the maximum of

$$\left| \int_a^b \int_a^b K(x,t)\varphi(x)\varphi(t)dxdt \right|$$

provided that

$$\int_a^b \varphi^2(x)dx = 1$$

if $K(x,t) = xt + x^2t^2,\ -1 \le x, t \le 1$;

2. Find the maximum of

$$\left|\int_a^b\int_a^b K(x,t)\varphi(x)\varphi(t)dxdt\right|$$

provided that

$$\int_a^b \varphi^2(x)dx = 1$$

if $K(x,t) = \begin{cases}(x+1)t, & 0 \le x \le t,\\ (t+1)x, & t \le x \le 1.\end{cases}$

3. Find the bifurcation points of the zero solutions of the

integral equations:

$$\varphi(x) = \lambda\int_0^1 (3x-2)t\big(\varphi(t)+\varphi^3(t)\big)dt$$

Self-Assessment Answers:

Exercise (12a)

1. There are no real characteristic numbers and eigenfunctions
2. $\lambda_1 = \frac{1}{\pi}, \varphi_1(x) = \sin x$
3. $\lambda_1 = -\frac{2}{\pi}, \lambda_2 = \frac{2}{\pi},\ \varphi_1(x) = \sin x,\ \varphi_2(x) = \cos x$
4. There are no real characteristic numbers and eigenfunctions
5. $\lambda_1 = \lambda_2 = -3,\ \varphi(x) = x - 2x^2$
6. $\lambda_1 = \frac{1}{2}, \varphi_1(x) = \frac{5}{2}x + \frac{10}{3}x^2$
7. $\lambda_1 = \frac{1}{4}, \varphi_1(x) = \frac{3}{2}x + x^2$
8. $\lambda_1 = -\frac{e}{2}, \varphi_1(x) = \sinh x$
9. None
10. There are no real characteristic numbers and eigenfunctions

Exercise (12b)

1. $\lambda_0 = 1;\ \varphi_0 = e^x;\ \lambda_n = -n^2\pi^2;\ \varphi_n(x) = \sin n\pi x + n\pi\cos n\pi x$
2. $\lambda_n = -\frac{\mu_n^2}{3};\ \varphi_n(x) = \sin\mu_n x + \mu_n\cos\mu_n x$, where μ_n are roots of the equation $\mu - \frac{1}{\mu} = 2\cot\mu$
3. $\lambda_n = 4n^2 - 1;\ \varphi_n(x) = \sin 2n\,x\ (n = 1, 2, \cdots)$
4. $\lambda_n = \left(n+\frac{1}{2}\right)^2 - 1;\ \varphi_n(x) = \sin\left(n+\frac{1}{2}\right)x$
5. $\lambda_n = -\frac{1-\mu_n^2}{\sin};\ \varphi_n(x) = \sin\mu_n(\pi + x)(n = 1,2,\cdots)$, where μ_n are roots of the equation $\tan 2\pi\mu = -\mu\tan 1$

6. $\lambda_n = 1 - \mu_n^2$; $\varphi_n(x) = \sin\mu_n x + \mu_n \cos\mu_n x$, where μ_n are roots of the equation $2\cot\pi\mu = \mu - \frac{1}{\mu}$
7. $\lambda_n = -\frac{1+\mu_n^2}{2}$; $\varphi_n(x) = \sin\mu_n x + \mu_n \cos\mu_n x$, where μ_n are roots of the equation $2\cot\mu = \mu - \frac{1}{\mu}$
8. $\lambda_n = -1 - \mu_n^2$; $\varphi_n(x) = \sin\mu_n x$, where μ_n are roots of the equation $\tan\mu = \mu(\mu > 0)$
12. $(a)\frac{\pi^4}{90}$; $(b)\frac{\pi^2}{16} - \frac{1}{2}$; $(c)\frac{1+e^{-2}}{8}$
13. $(a)\varphi_1(x) = 1,\ \varphi_2(x) = 2x - 1$;
    $(b)\varphi_1(x) = x, \varphi_2(x) = x^2$;
    $(c)\varphi_1(x) = 1, \varphi_2(x) = x, \varphi_3(x) = 3x^2 - 1$

Exercise (12c)

2. $\lambda_0 = \frac{3}{2\pi^3}, \varphi_0(x) = 1$; $\lambda_n = (-1)^n \frac{n^2}{4\pi}, \varphi_n^{(1)}(x) = \cos nx, \varphi_n^{(2)}(x) = \sin nx$

Exercise (12d)

1. $\frac{2}{3}$; $\varphi(x) = \pm\sqrt{\frac{3}{2}}x$
2. 1; $\varphi(x) = \pm\sqrt{\frac{2}{e^2-1}}e^x$
3. There are no bifurcation points

12.9 SUGGESTED READINGS:

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- **Prof. P. Vijaya Laxmi**

LESSON- 13

SOLUTION OF HOMOGENEOUS INTEGRAL EQUATIONS WITH DEGENERATE KERNEL

OBJECTIVES:

Learn to solve Homogeneous Integral Equations with Degenerate Kernel.

STRUCTURE:

13.1 Integral Equation

13.2 Degenerate Kernel in Integral Equations

13.3 Homogeneous Fredholm Integral Equation with Degenerate Kernel/ Separable Kernel

13.4 Summary

13.5 Technical Terms

13.6 Self -Assessment Questions

13.7 Suggested Readings

13.1 INTEGRAL EQUATION:

An equation is an integral equation in which an unknown function appears under the integral sign.

Example: $\varphi(x) = \int_a^b K(x,t)\,\varphi(t)dt$

13.2 DEGENERATE KERNEL IN INTEGRAL EQUATIONS:

A degenerate kernel (also called a separable kernel) is a special type of kernel in an integral equation that can be expressed as a finite sum of products of functions of separate variables:

$$K(x,t) = [\sum_{k=1}^{n} a_k(x) b_k\,(t)]$$

13.3 HOMOGENEOUS FREDHOLM INTEGRAL EQUATIODEGENERATE KERNEL/ SEPARABLE KERNEL:

A homogeneous Fredholm integral equation of the second kind with a degenerate kernel is a special type of integral equation of the form:

$$\varphi(x) = \lambda \int_a^b K(x,t)\,\varphi(t)dt \qquad \text{(i)}$$

where

$\varphi(x)$ is a unknown function to solve for,

λ is a scalar parameter (real or complex),

$K(x,t)$ is a kernel (degenerate kernel),

Remarks: The number λ is not an eigenvalue since for $\lambda = 0$, (i) yield $y(x) = 0$, which is a zero solution.

Remarks: If the kernel $K(x,t)$ is continuous in the rectangle R: $a \leq x \leq b, a \leq t \leq b$ and the number a and b are finite, then to every eigenvalue λ there exists a finite number of linearly independent eigenfunctions.

13.3.1 Example 1: Solve the equation

$$\varphi(x) - \lambda \int_0^{\pi} (\cos^2 x \cos 2t + \cos^3 t \cos 3x)\, \varphi(t) dt = 0$$

Solution: The homogeneous integral equation is

$$\varphi(x) - \lambda \int_0^{\pi} (\cos^2 x \cos 2t + \cos^3 t \cos 3x)\, \varphi(t) dt = 0 \qquad (1)$$

and the given kernel $K(x,t) = \cos^2 x \cos 2t + \cos^3 t \cos 3x$ is a degenerate kernel

$$\varphi(x) = \lambda[\cos^2 x \int_0^{\pi} \cos 2t\, \varphi(t) dt + \cos 3x \int_0^{\pi} \cos^3 t\, \varphi(t) dt] \qquad (2)$$

$$\varphi(x) = \lambda[\cos^2 x C_1 + \cos 3x\, C_2] \qquad (3)$$

where $C_1 = \int_0^{\pi} \cos 2t\, \varphi(t) dt$ and

$$C_2 = \int_0^{\pi} \cos^3 t\, \varphi(t) dt \qquad (4)$$

Substituting equation (3) in equation (4), we get

$$C_1 = \int_0^{\pi} \cos 2t\, \lambda[\cos^2 t C_1 + \cos 3t\, C_2] dt$$

$$C_2 = \int_0^{\pi} \cos^3 t\, \lambda[\cos^2 t C_1 + \cos 3t\, C_2] dt$$

or

$$C_1 \left[1 - \lambda \int_0^{\pi} \cos 2t \cos^2 t dt\right] - C_2 \lambda \int_0^{\pi} \cos 2t \cos 3t\, dt = 0$$

$$-C_1 \lambda \int_0^{\pi} \cos^5 t\, dt + C_2 \left[1 - \lambda \int_0^{\pi} \cos 3t \cos^3 t dt\right] = 0$$

Evaluating the integrals, we obtain a linear system of homogeneous equations:

$\left(1-\frac{\lambda\pi}{4}\right)C_1 = 0, \left(1-\frac{\lambda\pi}{8}\right)C_2 = 0$ (5)

The determinant of the eigenvalues will be

$$\begin{vmatrix} \left(1-\frac{\lambda\pi}{4}\right) & 0 \\ 0 & \left(1-\frac{\lambda\pi}{8}\right) \end{vmatrix} = 0, \Rightarrow \lambda_1 = \frac{4}{\pi} \text{ and } \lambda_2 = \frac{8}{\pi}$$

The characteristic numbers of this equation are $\lambda_1 = \frac{4}{\pi}$ and $\lambda_2 = \frac{8}{\pi}$; the corresponding eigenfunctions are of the form

$\varphi(x) = C_1\cos^2 x$, if $\lambda_1 = \frac{4}{\pi}$,

$\varphi(x) = C_2 \cos 3x$, if $\lambda_2 = \frac{8}{\pi}$,

& $\varphi(x) = 0$, if $\lambda_1 \neq \frac{4}{\pi}$ and $\lambda_2 \neq \frac{8}{\pi}$,

where C_1 and C_2 are arbitrary constants. The last zero solution is obtained from the general solutions for $C_1 = 0$ and $C_2 = 0$.

13.3.2 Example 2:

Solve the homogeneous integral equation

$$\varphi(x) - \lambda \int_0^{\pi} \cos(x+t)\, \varphi(t) dt = 0.$$

Solution:

Given homogeneous integral equation is

$\varphi(x) = \lambda \int_0^{\pi} \cos(x+t)\, \varphi(t) dt$

$\varphi(x) = \lambda \int_0^{\pi} (\cos x \cos t - \sin x \sin t)\, \varphi(t) dt$ (1)

and the kernel $K(x,t) = \cos x \cos t - \sin x \sin t$ is a degenerate kernel

i.e., $\varphi(x) = \lambda[\cos x \int_0^{\pi} \cos t\, \varphi(t) dt - \sin x \int_0^{\pi} \sin t\, \varphi(t) dt]$ (2)

$\varphi(x) = \lambda \cos x\, C_1 - \lambda \sin x\, C_2,$ (3)

where $C_1 = \int_0^{\pi} \cos t\, \varphi(t) dt$ and

$C_2 = \int_0^{\pi} \sin t\, \varphi(t) dt$ (4)

substituting equation (3) in equation (4), we get

$$C_1 = \int_0^{\pi} \cos t\, (\lambda C_1 \cos t - \lambda C_2 \sin t) dt$$

$$C_2 = \int_0^{\pi} \sin t\,(\lambda C_1 \cos t - \lambda C_2 \sin t)dt$$

or

$C_1\left[1 - \lambda \int_0^{\pi} \cos^2 t\, dt\right] + C_2 \lambda \int_0^{\pi} \cos t \sin t\, dt = 0$

$-C_1 \lambda \int_0^{\pi} \sin t \cos t\, dt + C_2\left[1 + \lambda \int_0^{\pi} \sin^2 t\, dt\right] = 0.$

Evaluating the integrals, we obtain a linear system of homogeneous equations:

$\left(1 - \frac{\lambda\pi}{2}\right) C_1 = 0, \left(1 + \frac{\lambda\pi}{2}\right) C_2 = 0$ (5)

The determinant of the eigenvalues is

$$\begin{vmatrix} \left(1 - \frac{\lambda\pi}{2}\right) & 0 \\ 0 & \left(1 + \frac{\lambda\pi}{2}\right) \end{vmatrix} = 0, \Rightarrow \lambda = \pm\frac{2}{\pi}.$$

For $\lambda = +\frac{2}{\pi}$, $\left(1 - \frac{2}{\pi}\frac{\pi}{2}\right) C_1 = 0 \Rightarrow C_1$ can be any non-zero arbitrary constant

$\left(1 + \frac{2}{\pi}\frac{\pi}{2}\right) C_2 = 0 \Rightarrow C_2 = 0.$

For $\lambda = -\frac{2}{\pi}$, $\left(1 + \frac{2}{\pi}\frac{\pi}{2}\right) C_2 = 0 \Rightarrow C_1 = 0$

$\left(1 - \frac{2}{\pi}\frac{\pi}{2}\right) C_2 = 0 \Rightarrow C_2$ can be any non-zero arbitrary constant.

If $C_2 = 0$, and C_1 is arbitrary then, the eigenfunction will be

$\varphi(x) = C_1 \cos x$ (or) $\varphi(x) = C \cos x$.

Is $C_1 = 0$, and C_2 is arbitrary then the eigenfunction will be

$\varphi(x) = -C_2 \sin x$ (or) $\varphi(x) = -C \sin x$

The eigenfunctions are $\varphi(x) = C \cos x$ and $\varphi(x) = -C \sin x$, corresponding to the eigenvalues $\frac{2}{\pi}$ and $-\frac{2}{\pi}$.

13.4 SUMMARY:

This unit provided the fundamental idea of the integral equation in particular homogeneous Fredholm integral equations. Definition and calculations of degenerate kernel were discussed. The method to solve the homogeneous Fredholm integral equations by using degenerate kernel is explained thoroughly with the help of numerous examples. For better understanding of readers few examples and self-assessment problems related to degenerate kernel were included.

13.5 TECHNICAL TERMS:

Integral Equation: An equation is an integral equation in which an unknown function appears under the integral sign

Example: $\varphi(x) = \int_a^b K(x,t)\, \varphi(t) dt$

Degenerate Kernel: A degenerate kernel (also called a separable kernel) is a special type of kernel in an integral equation that can be expressed as a finite sum of products of functions of separate variables:

$$K(x,t) = [\sum_{k=1}^{n} a_k(x) b_k(t)]$$

Homogeneous Fredholm Integral Equation with Degenerate Kernel: A homogeneous Fredholm integral equation of the second kind with a degenerate kernel is a special type of integral equation of the form

$$\varphi(x) = \lambda \int_a^b K(x,t)\, \varphi(t) dt$$

13.6 SELF- ASSESSMENT QUESTIONS:

Solve the following homogeneous integral equations:

1. $\varphi(x) - \lambda \int_0^1 arc \cos x\, \varphi(t)\, dt = 0$
2. $\varphi(x) - 2 \int_0^{\frac{\pi}{4}} \frac{\varphi(t)}{1+\cos 2t}\, dt = 0$
3. $\varphi(x) - \frac{1}{4} \int_{-2}^{2} |\, x\, | \varphi(t)\, dt = 0$
4. $\varphi(x) + 6 \int_0^1 (x^2 - 2xt)\, \varphi(t)\, dt = 0$

Answers:

1. $\varphi(x) = \begin{cases} C\, arc \cos x, & \lambda = 1 \\ 0, & \lambda \neq 1 \end{cases}$
2. $\varphi(x) = C$
3. $\varphi(x) = C|x|$
4. $\varphi(x) = (x - x^2)$

13.7 SUGGESTED READINGS:

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5. Integral equations, Krishna Prakashan Media(P) Ltd., Meerut.

- **Dr. Vinutha Tummala**

LESSON- 14

NON-HOMOGENEOUS SYMMETRIC INTEGRAL EQUATIONS & FREDHOLM ALTERNATIVE

OBJECTIVES:

Learn to Non-Homogeneous Integral Equations and Fredholm-Alternative.

STRUCTURE:

14.1 INTRODUCTION

14.2 NON HOMOGENEOUS SYMMETRIC EQUATION

14.3 FREDHOLM ALTERNATIVE

14.4 SUMMARY

14.5 TECHNICAL TERMS

14.6 SELF-ASSESSMENT QUESTIONS

14.7 SUGGESTED READINGS

14.1 INTRODUCTION:

A non-homogeneous symmetric integral equation is a type of integral equation that includes a non-zero term outside the integral, which distinguishes it from a homogeneous equation. In mathematics, the **Fredholm alternative**, named after Ivar Fredholm, is one of Fredholm's theorems resulting in Fredholm theory.

14.2 NON HOMOGENEOUS SYMMETRIC EQUATION:

The non-homogeneous Fredholm integral equation of the second kind is

$$\varphi(x) - \lambda \int_a^b K(x,t)\, \varphi(t) dt = f(x) \qquad (1)$$

where $\varphi(x)$ is an unknown function

$K(x,t)$ is the symmetric kernel i.e., $K(x,t) = K(t,x)$

$f(x)$ is a known function

λ is a parameter

If $f(x)$ is continuous and the parameter λ does not match with any characteristic numbers λ_n $(n = 1,2,3, ...)$ of the corresponding homogeneous integral equation.

$$\varphi(x) - \lambda \int_a^b K(x,t)\, \varphi(t) dt = 0 \qquad (2)$$

then the equation (1) has a unique continuous solution.

If the kernel is symmetric, the solution can be expressed as

$\varphi(x) = f(x) - \lambda \sum_{n=1}^{\infty} \frac{a_n}{\lambda-\lambda_n} \varphi_n(x)$ (3)

$\varphi_n(x)$ are eigenfunctions of the homogeneous equation (2).

a_n are coefficients of the homogeneous equation,

$a_n = \int_a^b f(x)\, \varphi_n(x) dx$ (4)

The series on the right hand side of (3) converges absolutely and uniformly in the square

$a \le x, t \le b.$

But if the parameter λ coincides with the one of the characteristic numbers,say $\lambda = \lambda_k$, of index q (multiplicity of the number λ_k), then equation(1) will not, generally speaking, have any solutions. Solutions exist if and only if the q conditions are fulfilled: $(f, \varphi_m)= 0$ (or) $\int_a^b f(x)\, \varphi_m(x) dx = 0$, $(m = 1,2 \dots . q)$ (5)

that is, if the function $f(x)$ is orthogonal to all eigenfunctions belonging to the characteristic number λ_k. In this case (1) has an infinity of solutions which contain q arbitrary constants and are given by the formula

$\varphi(x) = f(x) - \lambda \sum_{n=q+1}^{\infty} \frac{a_n}{\lambda-\lambda_n} \varphi_n(x) + C_1\varphi_1(x) + C_2\varphi_2(x) + \dots + C_q\varphi_q(x)$ (6)

where $C_1, C_2, \dots . C_q$ are arbitrary constants.

In case of the degenerate kernel

$$K(x,t) = \sum_{k=1}^{m} a_k(x) b_k(t),$$

formulas (3) and (6) will contain finite sums in place of series in their right-hand members.

When the right-hand side of equation (1), i.e., the function $f(x)$, is orthogonal to all eigenfunctions $\varphi_n(x)$ of equation (2), the function itself will be a solution of equation (1):

$$\varphi(x) = f(x).$$

14.2.1 Example 1: Solve the equation

$$\varphi(x) - \lambda \int_0^1 K(x,t)\, \varphi(t) dt = x,$$

where $K(x,t) = \begin{cases} x(t-1), & if\ 0 \le x \le t \\ t(x-1), & if\ t \le x \le 1 \end{cases}$

Solution: Rewrite the equation as

$\varphi(x) = x + \lambda \left[\int_0^x t(x-1)\, \varphi(t) dt + \int_x^1 x(t-1)\, \varphi(t) dt \right]$ (1)

Consider the homogeneous integral equation as

$$\varphi(x) = \lambda\left[\int_0^x t(x-1)\,\varphi(t)dt + \int_x^1 x(t-1)\,\varphi(t)dt\right]$$

$$\varphi(x) = \lambda\left[(x-1)\int_0^x t\,\varphi(t)dt + x\int_x^1 (t-1)\,\varphi(t)dt\right] \quad (2)$$

Differentiate the equation (2) with respect to 'x' on both sides, we have

$$\varphi'(x) = \lambda\left[\int_0^x t\,\varphi(t)dt + x(x-1)\varphi(x) + \int_x^1 (t-1)\,\varphi(t)dt - x(x-1)\varphi(x)\right]$$

$$\varphi'(x) = \lambda\left[\int_0^x t\,\varphi(t)dt + \int_x^1 (t-1)\,\varphi(t)dt\right] \quad (3)$$

Again differentiate the equation (3) with respect to 'x' on both sides, we get

$$\varphi''(x) = \lambda[x\varphi(x) - (x-1)\varphi(x)]$$

$$\Rightarrow \varphi''(x) - \lambda\varphi(x) = 0 \quad (4)$$

with the boundary conditions $\varphi(0) = 0$, and $\varphi(1) = 0$ (5)

Let us consider the following cases:

(i) When $\lambda = 0$. Then equation (4) reduces to $\varphi''(x) = 0$. Its general solution is

$$\varphi(x) = C_1 x + C_2$$

$\varphi(0) = 0$ then $C_2 = 0$ and $\varphi(1) = 0$ then $C_1 = 0$

Thus $\varphi(x)$=0, which is not an eigenfunction corresponding to an eigenvalue $\lambda = 0$.

(ii) When λ is positive i.e., $\lambda = \mu^2, \mu \neq 0$

The differential equation (4) is reduced to $\varphi''(x) - \mu^2\varphi(x) = 0$

then $\varphi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$

Since $\varphi(0) = 0 \Rightarrow C_1 + C_2 = 0$ and $\varphi(1) = 0 \Rightarrow C_1 e^{\mu} + C_2 e^{-\mu} = 0$

Then we get $C_1 = 0 = C_2$

Hence $\varphi(x)$=0, which is not an eigenfunction corresponding to an eigenvalue $\lambda > 0$.

(iii) When λ is negative i.e., $\lambda = -\mu^2, \mu \neq 0$

The differential equation (4) is reduced to $\varphi''(x) + \mu^2\varphi(x) = 0$

Whose solution is given by

$$\varphi(x) = C_1 \cos \mu x + C_2 \sin \mu x$$

Since $\varphi(0) = 0 \Rightarrow C_1 = 0$ and $\varphi(1) = 0 \Rightarrow C_2 \sin \mu$=0 $\Rightarrow \mu = n\pi, C_2 \neq 0,$

where n is any integer

Thus the required eigenvalue is given as $\lambda = -n^2\pi^2, n = 1,2,3 \ldots$

The corresponding eigenfunctions $\varphi_n(x)$ are given by

$\varphi_n(x) = \sin n\pi x, n = 1,2,3 \ldots; C_2 = 1$(let).

The normalized eigenfunctions $\Phi_n(x)$ are given by

$$\Phi_n(x) = \frac{\varphi_n(x)}{\{\int_0^1 [\varphi_n(x)]^2 dx\}^{1/2}} = \frac{\sin n\pi x}{\{\int_0^1 \sin^2 n\pi x \, dx\}^{1/2}} = \sqrt{2}\sin n\pi x.$$

Hence

$$F_n = \int_0^1 F(x)\,\Phi_n(x)\,dx$$

$$\Rightarrow F_n = \int_0^1 x.(\sqrt{2}\sin n\pi x.)\,dx$$

$$\Rightarrow F_n = \sqrt{2}\left\{-\left(\frac{x\cos n\pi x}{n\pi}\right)_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx\right\}$$

$$\Rightarrow F_n = \sqrt{2}\left\{-\frac{(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}(\sin n\pi x)_0^1\right\} = \frac{(-1)^{n+1}\sqrt{2}}{n\pi}$$

The given integral equation contains a unique solution as

$$\varphi(x) = F(x) + \lambda\sum_{n=1}^{\infty}\frac{F_n}{\lambda_n - \lambda}.\Phi_n(x), \lambda \neq \lambda_n$$

$$\Rightarrow \varphi(x) = x + \lambda\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sqrt{2}}{n\pi}.\frac{\sqrt{2}\sin n\pi x}{-n^2\pi^2 - \lambda}$$

$$\Rightarrow \varphi(x) = x + \frac{2\lambda}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n(n^2\pi^2 + \lambda)}.$$

Again, when $\lambda = \lambda_n = -n^2\pi^2, n = 1,2,3, \ldots$ then the integral equation does not possess any solution.

14.2.2 Example 2: Solve the equation

$$\varphi(x) - \lambda\int_0^1 K(x,t)\,\varphi(t)dt = \cos\pi x,$$

where $K(x,t) = \begin{cases}(x+1)t, & 0 \leq x \leq t \\ (t+1)x, & t \leq x \leq 1\end{cases}$

Solution: Rewrite the equation as

$$\varphi(x) = \cos \pi x + \lambda \left[\int_0^x (t+1)x\,\varphi(t)dt + \int_x^1 (x+1)t\,\varphi(t)dt\right] \quad (1)$$

Consider the homogeneous integral equation as

$$\varphi(x) = \lambda \left[\int_0^x (t+1)x\,\varphi(t)dt + \int_x^1 (x+1)t\,\varphi(t)dt\right]$$

$$\varphi(x) = \lambda \left[x\int_0^x (t+1)\,\varphi(t)dt + (x+1)\int_x^1 t\,\varphi(t)dt\right] \quad (2)$$

Differentiate the equation (2) with respect to 'x' on both sides, we have

$$\varphi'(x) = \lambda \left[\int_0^x (t+1)\,\varphi(t)dt + x(x+1)\varphi(x) + \int_x^1 t\,\varphi(t)dt - x(x+1)\varphi(x)\right]$$

$$\varphi'(x) = \lambda \left[\int_0^x (t+1)\varphi(t)dt + \int_x^1 t\,\varphi(t)dt\right] \quad (3)$$

Again differentiate the equation (3) with respect to 'x' on both sides, we get

$$\varphi''(x) = \lambda[(x+1)\varphi(x) - x\varphi(x)]$$

$$\Rightarrow \varphi''(x) - \lambda\varphi(x) = 0 \quad (4)$$

The characteristic numbers are $\lambda_0 = 1, \lambda_n = -n^2\pi^2$, then the solution of the given equation will have the form $\varphi_0(x) = e^x, \varphi_n(x) = \sin n\pi x + n\pi \cos n\pi x$ $(n = 1,2, \ldots)$

If $\lambda \neq 1$ and $\lambda \neq -n^2\pi^2$, then the solution of the given equation will have the form

$$\varphi(x) = cos\ \pi x - \lambda \left[\frac{a_0 e^x}{\lambda - 1} + \sum_{n=1}^{\infty} \frac{a_n}{\lambda + n^2\pi^2}(\sin n\pi x + n\pi \cos n\pi x)\right]$$

and since

$$a_0 = \int_0^1 e^x \cos \pi x dx = -\frac{1+e}{1+\pi^2}$$

$$a_{n=} \int_0^1 \cos \pi x\,(\sin n\pi x + n\pi \cos n\pi x)dx = \begin{cases} 0, n \neq 1, \\ \frac{\pi}{2}, n = 1 \end{cases}$$

it follows that

$$\varphi(x) = \cos \pi x + \lambda \left[\frac{1+e}{1+\pi^2}\frac{e^x}{\lambda - 1} - \frac{\pi}{2(\lambda + \pi^2)}(\sin \pi x + \pi \cos \pi x)\right]$$

For $\lambda = 1$ and $\lambda = -\pi^2 (n = 1)$ the equation has no solutions since its right-hand side, that is, the function $\cos \pi x$, is not orthogonal to the corresponding eigenfunctions

$$\varphi_0(x) = e^{x,}$$

$$\varphi_1(x) = \sin \pi x + \pi \cos \pi x$$

But if $\lambda = -n^2\pi^2$, where $n = 2,3, ...$, then the given equation has an infinity of solutions which are given by formula (6):

$$\varphi(x) = \cos\pi x + \lambda\left[\frac{1+e}{1+\pi^2}\frac{e^x}{\lambda-1} - \frac{\pi}{2(\lambda+\pi^2)}(\sin\pi x + \pi\cos\pi x)\right] + C(\sin\pi x + \pi\cos\pi x)$$

where C is an arbitrary constant.

In certain cases, a nonhomogeneous symmetric integral equation can be reduced to a nonhomogeneous boundary-value problem. This is possible when the kernel $K(x,t)$ of the integral equation is a Green's function of some linear differential operator.

14.2.3 Example 3: Solve the equation

$$\varphi(x) - \lambda\int_0^1 K(x,t)\,\varphi(t)dt = e^x,$$

where $K(x,t) = \begin{cases} \frac{\sinh x\sinh(t-1)}{\sinh 1}, & 0 \le x \le t, \\ \frac{\sinh t\sinh(x-1)}{\sinh 1}, & t \le x \le 1. \end{cases}$

Solution: Rewrite the equation as

$$\varphi(x) = e^x + \lambda\int_0^1 K(x,t)\,\varphi(t)dt \quad (1)$$

$$\varphi(x) = e^x + \lambda\int_0^x K(x,t)\,\varphi(t)dt + \lambda\int_x^1 K(x,t)\,\varphi(t)dt$$

$$\varphi(x) = e^x + \lambda\int_0^x \frac{\sinh t\sin\ (x-1)}{\sinh}\varphi(t)dt + \lambda\int_x^1 \frac{\sinh x\sinh(t-1)}{\text{si}}\varphi(t)dt$$

$$\varphi(x) = e^x + \lambda\frac{\sinh(x-1)}{\sinh 1}\int_0^x \sinh t\,\varphi(t)dt + \lambda\frac{\sinh x}{\sinh 1}\int_x^1 \sinh(t-1)\,\varphi(t)dt \quad (2)$$

The boundary conditions of equation (2) as

$$\varphi(0) = 1, \varphi(1) = e \quad (3)$$

Differentiate the equation (2) with respect to x on both sides, we have

$$\varphi'(x) = e^x + \lambda\frac{\cosh(x-1)}{\sinh 1}\int_0^x \sinh t\,\varphi(t)dt + \frac{\lambda\sinh(x-1)}{\sinh 1}\sinh x\varphi(x) + \lambda\frac{\cosh x}{\text{si}}\int_x^1 \sinh(t-1)\,\varphi(t)dt - \frac{\lambda\sin\ (x-1)}{\sinh}\sinh x\varphi(x)$$

$$\varphi'(x) = e^x + \lambda\frac{\cosh(x-1)}{\sinh 1}\int_0^x \sinh t\,\varphi(t)dt + \lambda\frac{\cosh x}{\sinh 1}\int_x^1 \sinh(t-1)\,\varphi(t)dt \quad (4)$$

Differentiate the equation (4) with respect to x on both sides, we have

$\varphi''(x) = e^x + \lambda \frac{\sinh(x-1)}{\sinh 1} \int_0^x \sinh t\, \varphi(t) dt + \lambda \frac{\cosh (x-1)}{\sinh 1} \sinh x\, \varphi(x) + \lambda \frac{\sinh x}{\sinh 1} \int_x^1 \sinh(t - 1)\, \varphi(t) dt - \lambda \frac{\cosh x}{\sin} \sinh(x-1)\, \varphi(x)$

$\varphi''(x) = e^x + \lambda \frac{\sinh(x-1)}{\sinh 1} \int_0^x \sin ht\, \varphi(t) dt + \lambda \frac{\sinh x}{\sinh 1} \int_x^1 \sinh(t-1)\, \varphi(t) dt + \frac{\lambda}{\mathrm{si}} \varphi(x)[\cosh (x-1) \sinh x - \cosh x \sinh(x-1)]$

$\varphi''(x) = e^x + \varphi(x) + \lambda\varphi(x)$

$$\varphi''(x) - (\lambda + 1)\varphi(x) = e^x \quad (5)$$

Let us consider the following cases:

(i) $(\lambda + 1) = 0$, or $\lambda = -1$

Equation (5) is of the form

$$\varphi(x) = C_1 x + C_2 + e^x$$

Taking into the boundary conditions (3), we get the following system for finding the constants C_1 and C_2

$$\begin{cases} C_2 + 1 = 1, \\ C_1 + C_2 + e = e \end{cases}$$

Its solution is of the form $C_1 = 0, C_2 = 0$, and, hence,

$$\varphi(x) = e^x$$

(ii) $(\lambda + 1) > 0$, or $\lambda > -1, \lambda \neq 0$. The general solution of equation (5) is $\varphi(x) = C_1 \cosh\sqrt{(1+\lambda)}x + C_2 \sinh\sqrt{(1+\lambda)}x - \frac{e^x}{\lambda}$

The boundary conditions (3) yield the following system for finding C_1 and C_2:

$$\begin{cases} C_1 - \frac{1}{\lambda} = 1, \\ C_1 \cosh\sqrt{(1+\lambda)} + C_2 \sinh\sqrt{(1+\lambda)} - \frac{e}{\lambda} = e \end{cases}$$

whence

$$C_1 = \left(1 + \frac{1}{\lambda}\right), C_2 = \frac{e - \cosh\sqrt{(1+\lambda)}}{\sinh\sqrt{(1+\lambda)}}\left(1 + \frac{1}{\lambda}\right)$$

Then the general solution is

$\varphi(x) = \left(1+\frac{1}{\lambda}\right).\cosh\sqrt{(1+\lambda)}x + \frac{e-\cosh\sqrt{(1+\lambda)}}{\sinh\sqrt{(1+\lambda)}}\left(1+\frac{1}{\lambda}\right).\sinh\sqrt{(1+\lambda)}x - \frac{e^x}{\lambda}$

$\varphi(x) = \left(1+\frac{1}{\lambda}\right).\frac{\sinh\sqrt{(1+\lambda)}.(1-x)}{\sinh\sqrt{(1+\lambda)}} - \frac{e^x}{\lambda}.$

(iii) $(\lambda+1) < 0$, or $\lambda < -1, \lambda+1 = -\mu^2$. The general solution of equation (5) is

$$\varphi(x) = C_1\cos\mu x + C_2\sin\mu x + \frac{e^x}{1+\mu^2}$$

The boundary conditions (3) yield the following system for finding C_1 and C_2:

$$\left.\begin{aligned} C_1 + \frac{1}{1+\mu^2} &= 1, \\ C_1\cos\mu + C_2\sin\mu &= e\frac{\mu^2}{1+\mu^2} \end{aligned}\right\} \qquad (6)$$

In turn, two cases are possible here:

(a) μ is not a root of the equation $\sin\mu = 0$.

$C_1 = \frac{\mu^2}{1+\mu^2}, C_2 = \frac{(e-\cos\ )\mu^2}{(1+\mu^2)\sin\mu}$

and, hence,

$\varphi(x) = \frac{\mu^2}{1+\mu^2}\left[\cos\mu x + \frac{e-\cos\mu}{\sin\mu}\sin\mu x\right] + \frac{e^x}{1+\mu^2}.$

where $\mu = \sqrt{-\lambda-1}.$

(b) μ is a root of the equation $\sin\mu = 0$, i.e., $\mu = n\pi\ (n = 1,2,\ldots)$. System (6) is inconsistence and, consequently, the given equation (1) has no solutions.

In this case, the corresponding homogeneous integral equation

$\varphi(x) + (1+n^2\pi^2)\int_0^1 K(x,t)\ \varphi(t)dt = 0$ (7)

will have an infinity of nontrivial solutions, that is, the number $\lambda_n = -(1+n^2\pi^2)$ are characteristic numbers and their associated solutions $\varphi_n(x) = \sin n\pi x$ are eigenfunctions of equation (7).

14.2.4 Problem: Solve the equation

$$\varphi(x) - \frac{\pi^2}{4}\int_0^1 K(x,t)\,\varphi(t)dt = \frac{x}{2},$$

$$K(x,t) = \begin{cases} \frac{x(2-t)}{2}, & 0 \le x \le t \\ \frac{t(2-x)}{2}, & t \le x \le 1 \end{cases}.$$

Solution: The equation can be written as

$$\varphi(x) = \frac{x}{2} + \frac{\pi^2}{4}\left[\int_0^x \frac{t(2-x)}{2}\varphi(t)dt + \int_x^1 \frac{x(2-t)}{2}\varphi(t)dt\right] \quad (1)$$

Differentiating the equation (1) with respect to 'x' on both sides, we get

$$\varphi'(x) = \frac{1}{2} + \frac{\pi^2}{4}(2-x)\varphi(x) - \frac{\pi^2}{4}\left[\int_0^x \frac{t}{2}\varphi(t)dt + \int_x^1 \frac{x}{2}\varphi(t)dt\right].$$

$$\varphi'(x) = \frac{1}{2} + \frac{\pi^2}{4}\left[-\int_0^x \frac{t}{2}\varphi(t)dt + \frac{x(2-t)}{2}\varphi(x) + \int_x^1 \frac{2-t}{2}\varphi(t)dt - \frac{x(2-t)}{2}\varphi(x)\right]$$

$$\varphi'(x) = \frac{1}{2} + \frac{\pi^2}{4}\left[-\int_0^x \frac{t}{2}\varphi(t)dt + \int_x^1 \frac{2-t}{2}\varphi(t)dt\right] \quad (2)$$

Again differentiating the equation (2) with respect to 'x' on both sides, we get

$$\varphi''(x) = \frac{\pi^2}{4}\left[-\frac{x}{2}\varphi(x) - \frac{2-x}{2}\varphi(x)\right]$$

$$\varphi''(x) = \frac{\pi^2}{4}\left[-\frac{x}{2}\varphi(x) - \varphi(x) + \frac{x}{2}\varphi(x)\right]$$

$$\varphi''(x) + \frac{\pi^2}{4}[\varphi(x)] = 0. \quad (3)$$

The general solution of the above differential equation (3) is

$$\varphi(x) = C_1 \cos\frac{\pi x}{2} + C_2 \sin\frac{\pi x}{2} \quad (4)$$

and the boundary conditions are

$$\varphi(0) = C_1 \cos 0 + C_2 \sin 0 \Rightarrow C_1 = 0$$

$$\varphi(1) = C_1 \cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2} \Rightarrow C_2 = 1$$

The required function $\varphi(x)$ which is a solution of the non homogeneous boundary value problem is

$$\varphi(x) = \sin\frac{\pi x}{2}.$$

14.3 FREDHOLM ALTERNATIVE:

In mathematics, the **Fredholm alternative**, named after Ivar Fredholm, is one of Fredholm's theorems resulting in Fredholm theory. It may be expressed in several ways, as a theorem of linear algebra, a theorem of integral equations, or as a theorem on Fredholm operators. Part of the result states that a non-zero complex number in the spectrum of a compact operator is an eigen value.

14.3.1 Theorem 1 (Fredholm Alternative):

Either the non-homogeneous linear equation of the second kind

$$\varphi(x) - \lambda \int_a^b K(x,t)\,\varphi(t)dt = f(x). \qquad (1)$$

has a unique solution for any function $f(x)$ (in some sufficiently broad class) or the corresponding homogeneous equation

$$\varphi(x) - \lambda \int_a^b K(x,t)\,\varphi(t)dt = 0. \qquad (2)$$

has at least one nontrivial (that is, not identically zero) solution.

14.3.2 Theorem 2:

If the first alternative holds true for equation (1), then it holds true for the associated equation

$$\Psi(x) - \lambda \int_a^b K(t,x)\,\Psi(t)dt = g(x). \qquad (3)$$

as well. The homogeneous integral equation (2) and its associated equation

$$\Psi(x) - \lambda \int_a^b K(t,x)\,\Psi(t)dt = 0. \qquad (4)$$

have one and the same finite number of linearly independent solutions.

14.3.2.1 Note:

If the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$, are solutions of the homogeneous equation (2), then their linear combination

$$\varphi(x) = C_1\varphi_1(x), + C_2\varphi_2(x) + \cdots + C_n\varphi_n(x) = \sum_{k=1}^{n} C_k\,\varphi_k(x)$$

where the $C_k (k = 1,2, \ldots, n)$ are arbitrary constants, is also a solution of the equation.

14.3.3 Theorem 3:

A necessary and sufficient condition for the existence of a solution $\varphi(x)$ of the non-homogeneous equation (1) in the latter case of all the alternative is the condition of orthogonality of the right side of the equation, i.e., of the function $f(x)$, to any solutions $\Psi(x)$ of the homogeneous equation (4) associated with equation (2):

$$\int_a^b f(x)\,\Psi(x)dx = 0 \qquad (5)$$

14.3.3.1 Note:

When condition (5) is fulfilled, equation (1) will have an infinite number of solutions, since this equation will be satisfied by any function of the form $\varphi(x) + \bar{\varphi}(x)$, where $\varphi(x)$ is some solution of equation (1) and $\bar{\varphi}(x)$ is any solution of the corresponding homogeneous equation (2). Besides, if equation (1) is satisfied by the function $\varphi_1(x)$ and $\varphi_2(x)$, then by virtue of the linearity of the equation the difference $\varphi_1(x) - \varphi_2(x)$, is a solution of the corresponding homogeneous equation (2).

The Fredholm alternative is particularly important in practical situations. Instead of proving that a given integral equation (1) has a solution, it is often simpler to prove that the appropriate homogeneous equation (2) or its associated equation (4) has only trivial solutions. Whence it follows, by virtue of the alternative, that equation (1) indeed has a solution.

14.3.3.2 Remarks:

(1) If the kernel $K\,(x,t)$ of the integral equation (1) is a symmetric, that is, $K(x,t) = K\,(t,x)$, then the associated homogeneous equation (4) coincides with the homogeneous equation (2) which corresponds to equation (1).

(2) In the case of non-homogeneous integral equation with a degenerate kernel

$$\varphi(x) - \lambda \int_a^b \left[\sum_{k=1}^{n} a_k(x) b_k\,(t)\right] \varphi(t) dt = f(x)$$

the orthogonality condition (5) of the right side of this equation yields n equalities

$\int_a^b f(t) b_k(t) dt = 0\ (k = 1,2, \dots, n)$

14.3.4 Example 1: Solve

$$\varphi(x) - \lambda \int_0^1 (5x^2 - 3)t^2\, \varphi(t) dt = e^x$$

Solution: Given integral equation is

$\varphi(x) - \lambda \int_0^1 (5x^2 - 3)t^2\, \varphi(t)\, dt = e^x$

From which, we have

$\varphi(x) = C\lambda(5x^2 - 3) + e^x$, (1)

where $C = \int_0^1 t^2\, \varphi(t)\, dt$ (2)

From (1) and (2), we get

$$C = \int_0^1 t^2 [C\, \lambda(5t^2 - 3) + e^t]\, dt$$

$C = \int_0^1 C\, \lambda(5t^4 - 3t^2)\,]\, dt + \int_0^1 t^2 e^t\, dt$

Whence, $C = e - 2$

For any λ, the given equation has a unique solution:

$\varphi(x) = (e - 2)\lambda(5x^2 - 3) + e^x$

and the corresponding homogeneous equation

$$\varphi(x) - \lambda \int_0^1 (5x^2 - 3)t^2 \, \varphi(t) \, dt = 0$$

has a unique zero solution $\varphi(x) = 0$.

14.3.5 Example 2: Solve $\varphi(x) - \lambda \int_0^1 \sin \ln x \, \varphi(t) \, dt = 2x$

Solution:

We have, $\varphi(x) = 2x + \lambda \int_0^1 \sin \ln x \, \varphi(t) \, dt$

$$\varphi(x) = 2x + \lambda \sin \ln x \, C$$

where $C = \int_0^1 \varphi(t) \, dt$. Substituting the expression $\varphi(t)$ into the integral, we obtain

$C = C\lambda \int_0^1 \sin \ln t \, dt$+1

whence $C\left(1 + \frac{\lambda}{2}\right) = 1$

If $\lambda \neq -2$, then the given equation has a unique solution
$\varphi(x) = 2x + \frac{2\lambda}{2+\lambda} \sin \ln x$; the corresponding homogeneous equation $\varphi(x) - \lambda \int_0^1 \sin \ln x \, \varphi(t) \, dt = 0$

has only the zero solution $\varphi(x) \equiv 0$.

But if $\lambda = -2$, then the given equation does not have any solutions since the right side $f(x) = 2x$ is not orthogonal to the function $\sin \ln x$; the homogeneous equation has an infinity of solutions since it follows from the equation defining C, $0.C = 0$, that C is an arbitrary constant; all these solutions are given by the formula $\varphi(x) = \widetilde{C} \sin \ln x$ ($\widetilde{C} = -2C$)

14.3.6 Example 3: Solve

$\varphi(x) - \lambda \int_0^\pi \cos(x+t) \, \varphi(t) \, dt = \cos 3x$

Solution: Rewrite the equation in the form

$$\varphi(x) - \lambda \int_0^\pi (\cos x \cos t - \sin x \sin t) \, \varphi(t) \, dt = \cos 3x$$

Whence we have

$$\varphi(x) = C_1 \lambda \cos x - C_2 \lambda \sin x + \cos 3x \qquad (1)$$

$$\text{where} \begin{cases} C_1 = \int_0^\pi \varphi(t) \cos t \, dt, \\ C_2 = \int_0^\pi \varphi(t) \sin t \, dt \end{cases} \qquad (2)$$

Substituting (1) into (2), we get

$$\begin{cases} C_1 = \int_0^{\pi} (C_1\lambda \cos t - C_2\lambda \sin t + \cos 3t) \cos t\, dt, \\ C_2 = \int_0^{\pi} (C_1\lambda \cos t - C_2\lambda \sin t + \cos 3t) \sin t\, dt \end{cases}$$

whence

$$\begin{cases} C_1(1 - \lambda \int_0^{\pi} \cos^2 t\, dt) + C_2\lambda \int_0^{\pi} \sin t \cos t\, dt = \int_0^{\pi} \cos 3t \cos t\, dt, \\ - C_1\lambda \int_0^{\pi} \cos t \sin t\, dt + C_2\,(1 + \lambda \int_0^{\pi} sin^2\, t\, dt) = \int_0^{\pi} \cos 3t \sin t\, dt, \end{cases} \quad \text{or}$$

$$\begin{cases} C_1\left(1 - \lambda\frac{\pi}{2}\right) = 0, \\ C_2\left(1 + \lambda\frac{\pi}{2}\right) = 0 \end{cases} \tag{3}$$

The determinant of the system is

$$\Delta(\lambda) \begin{vmatrix} 1 - \lambda\dfrac{\pi}{2} & 0 \\ 0 & 1 + \lambda\dfrac{\pi}{2} \end{vmatrix} = 1 - \lambda^2\frac{\pi^2}{4}$$

(i) If $\lambda \neq \pm\frac{2}{\pi}\,(\Delta(\lambda) \neq 0)$, then system (3) has a unique solution $C_1 = 0, C_2 = 0$ and, hence, the given equation has the unique solution $\varphi(x) = \cos 3x$ and the corresponding homogeneous equation

$$\varphi(x) - \lambda \int_0^{\pi} \cos(x + t)\,\varphi(t)\, dt = 0 \tag{4}$$

only has the zero solution $\varphi(x) = 0$.

(ii) If $\lambda = \frac{2}{\pi}$, then system (3) takes the form $\begin{cases} C_1.\,0 = 0, \\ C_2.\,2 = 0 \end{cases}$

Whence it follows that $C_2 = 0$ and $C_1 = C$, where C is an arbitrary constant. The given equation will have an infinity of solution which are given by the formula

$$\varphi(x) = \frac{2}{\pi}C \cos x + \cos 3x$$

or

$$\varphi(x) = \tilde{C} \cos x + \cos 3x, \left(\tilde{C} = \frac{2C}{\pi}\right);$$

the corresponding homogeneous equation (4) has an infinity of solutions:

$$\varphi(x) = \tilde{C} \cos x$$

(iii) If $\lambda = -\frac{2}{\pi}$, then system (3) takes the form

$$\begin{cases} 2.\,C_1 = 0, \\ 0.\,C_2 = 0 \end{cases}$$

whence $C_1 = 0, C_2 = C$, where C is an arbitrary constant. The general solution of the given equation is of the form

$$\varphi(x) = \tilde{C}\sin x + \cos 3x \left(\tilde{C} = \frac{2C}{\pi}\right);$$

In this example, the kernel $K(x,t) = \cos(x+t)$ of the given equation is symmetric: $K(x,t) = K(t,x)$; the right side of the equation [that is, the function $f(x) = \cos 3x$] is orthogonal to the functions $\cos x$ and $\sin x$ on the interval $[0,\pi]$.

14.3.7 Example 4:

Solve $\varphi(x) - \lambda \int_0^\pi \cos^2(x)\,\varphi(t)\,dt = 1$

Solution:

We have, $\varphi(x) = 1 + \lambda\cos^2 x \int_0^\pi \varphi(t)\;dt$ (1)

$\Rightarrow \varphi(x) = 1 + \lambda\cos^2 xC$ (2)

where $C=\int_0^\pi \varphi(t)\;dt.$ (3)

Substituting (2) into (3), we get

$C=\int_0^\pi (1 + \lambda\cos^2 tC)\;dt$

$C=\int_0^\pi dt + \frac{\lambda C}{2}\int_0^\pi (1 + \cos 2t)\,dt$

Whence $C=\frac{2\pi}{(2-\lambda\pi)}$

Case (i): If $\lambda \neq \frac{2}{\pi}$, then the given equation has a unique solution given by

$$\varphi(x) = 1 + \frac{2\pi\lambda}{(2-\lambda\pi)}\cos^2 x$$

and the corresponding homogeneous equation

$$\varphi(x) - \lambda\int_0^\pi \cos^2(x)\,\varphi(t)\,dt = 0$$

has the zero solution, $\varphi(x) = 0$.

Case (ii): If $\lambda = \frac{2}{\pi}$, then the given equation does not has any solution.

$$\therefore\ \varphi(x) = \begin{cases} 1 + \dfrac{2\pi\lambda}{(2-\lambda\pi)}\cos^2 x, & \lambda \neq \dfrac{2}{\pi} \\ \text{has no solution}, & \lambda = \dfrac{2}{\pi} \end{cases}.$$

14.4 SUMMARY:

This chapter provided the basic idea of the non-homogeneous symmetric equation and a theorem namely Fredholm alternative along with its proof. Few more theorems and their proofs related to this topic were given in detail with appropriate examples. By making use of these results and example problems the reader will get a clear idea on how to solve non-homogeneous symmetric integral equations. Finally same self-assessment problems were provided for practice.

14.5 TECHNICAL TERMS:

Non Homogeneous Symmetric Equation:
The non-homogeneous Fredholm integral equation of the second kind is

$$\varphi(x) - \lambda \int_a^b K(x,t)\,\varphi(t)dt = f(x).$$

where $\varphi(x)$ is an unknown function

$K(x,t)$ is the symmetric kernel i.e., $K(x,t) = K(t,x)$

$f(x)$ is a known function

λ is a parameter

Fredholm Alternative Theorem:
Either the non-homogeneous linear equation of the second kind

$$\varphi(x) - \lambda \int_a^b K(x,t)\,\varphi(t)dt = f(x). \quad (1)$$

has a unique solution for any function $f(x)$ (in some sufficiently broad class) or the corresponding homogeneous equation

$$\varphi(x) - \lambda \int_a^b K(x,t)\,\varphi(t)dt = 0. \quad (2)$$

has at least one nontrivial (that is, not identically zero) solution.

14.6 SELF-ASSESSMENT QUESTIONS:

Section-A:
Solve the following homogeneous symmetric integral equations:

1. $\varphi(x) + \int_0^1 K(x,t)\,\varphi(t)\,dt = xe^x,$

$$K(x,t) = \begin{cases} \frac{\sinh x \sinh(t-1)}{\sinh 1}, & 0 \le x \le t, \\ \frac{\sinh t \sinh(x-1)}{\sinh 1}, & t \le x \le 1. \end{cases}$$

2. $\varphi(x) - \lambda \int_0^1 K(x,t)\,\varphi(t)\,dt = x - 1,$

$$K(x,t) = \begin{cases} x - t, & 0 \le x \le t, \\ t - x, & t \le x \le 1. \end{cases}$$

3. $\varphi(x) - 2\int_0^{\frac{\pi}{2}} K(x,t)\,\varphi(t)\,dt = \cos 2x,$

$$K(x,t) = \begin{cases} \sin x\ \cos t, & 0 \le x \le t, \\ \sin t \cos x, & t \le x \le \frac{\pi}{2}. \end{cases}$$

4. $\varphi(x) - \lambda\int_0^{\pi} K(x,t)\,\varphi(t)\,dt = 1,$

$$K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x \le t, \\ \sin t \cos x, & t \le x \le \pi. \end{cases}$$

5. $\varphi(x) - \lambda\int_0^{1} K(x,t)\,\varphi(t)\,dt = x,$

$$K(x,t) = \begin{cases} (x+1)(t-3), & 0 \le x \le t, \\ (t+1)(x-3), & t \le x \le 1. \end{cases}$$

6. $\varphi(x) - \lambda\int_0^{\pi} K(x,t)\,\varphi(t)\,dt = \sin x,$

$$K(x,t) = \begin{cases} \sin(x+\frac{\pi}{4})\sin(t-\frac{\pi}{4}), & 0 \le x \le t, \\ \sin(t+\frac{\pi}{4})\sin(x-\frac{\pi}{4}), & t \le x \le \pi. \end{cases}$$

7. $\varphi(x) - \lambda\int_0^{1} K(x,t)\,\varphi(t)\,dt = \sinh x,$

$$K(x,t) = \begin{cases} -e^{-t}\sinh x, & 0 \le x \le t, \\ -e^{-x}\sinh t, & t \le x \le 1 \end{cases}$$

8. $\varphi(x) - \lambda\int_0^{1} K(x,t)\,\varphi(t)\,dt = \cosh x,$

$$K(x,t) = \begin{cases} \frac{\cosh x \cosh (t-1)}{\sinh}, & 0 \le x \le t, \\ \frac{\cosh t \cosh (x-1)}{\sinh}, & t \le x \le 1. \end{cases}$$

9. $\varphi(x) - \lambda\int_0^{\pi} |\,x - t\,|\,\varphi(t)\ dt = 1.$

Section-B

Investigate the solvability of the following integral equations (for different values of the parameter λ):

1. $\varphi(x) - \lambda\int_{-1}^{1} xe^{t}\,\varphi(t)\,dt = x.$

2. $\varphi(x) - \lambda\int_0^{2\pi} |x-\pi|\,\varphi(t)\,dt = x$

3. $\varphi(x) - \lambda\int_0^{1}(2xt - 4x^2)\,\varphi(t)\,dt = 1 - 2x$

4. $\varphi(x) - \lambda\int_{-1}^{1}(x^2 - 2xt)\,\varphi(t)\,dt = x^3 - x$

5. $\varphi(x) - \lambda\int_0^{2\pi}(\frac{1}{\pi}\cos x \cos t + \frac{1}{\pi}\sin 2x \sin 2t)\,\varphi(t)dt = \sin x$

6. $\varphi(x) - \lambda\int_0^{1} K(x,t)\,\varphi(t)\,dt = 1$

where

$$K(x,t) = \begin{cases} \cosh x . \sinh t, & 0 \le x \le t, \\ \cosh t . \sinh x, & t \le x \le 1. \end{cases}$$

Answers to Self- Assessment Questions:

Section-A:

1. $\varphi(x) = x - 2 + 2e^x$

2. $$\varphi(x) = \begin{cases} \frac{\sin \mu x + \sin \mu(x-1) - \mu \cos \mu x}{2\mu \cos\frac{\mu}{2}(\cos\frac{\mu}{2} + \frac{\mu}{2}\sin\frac{\mu}{2})}, \lambda > 0, \\ \frac{\sin \mu x + \sin \mu(x+1) - \mu \cos \mu x}{2\mu \cos\frac{\mu}{2}(\cos\frac{\mu}{2} - \sin\frac{\mu}{2})}, \lambda < 0, \\ \text{where } \mu = \sqrt{2\lambda} \end{cases}$$

3. $\varphi(x) = \cos 2\, x + 4\sum_{n=1}^{\infty} \frac{n \sin 2}{(4n^2-1)(4n^2-3)}$

4. $$\varphi(x) = \begin{cases} \frac{\lambda \cos\sqrt{\lambda+1}(\pi - x) + \cos\sqrt{\lambda+1}\,\pi}{(\lambda+1)\cos\pi\sqrt{\lambda+1}}, & \lambda > -1, \\ \frac{\lambda \cos h\sqrt{-\lambda-1}(\pi - x) + \cos h\sqrt{-\lambda-1}\,\pi}{(\lambda+1)\cos\pi\sqrt{-\lambda-1}}, & \lambda < -1, \\ \frac{x^2}{2} - \pi x + 1, & \lambda = -1 \end{cases}$$

5. $$\varphi(x) = \begin{cases} \frac{3(\sin \text{h}\, \mu + \mu \cosh \mu x) + \sinh \quad (x-1) - 2\mu \cosh \mu(x-1)}{(1+2\mu^2)\sin h\mu + 3\mu \text{ c}}, \\ \qquad \lambda > 0 (\mu = 2\sqrt{\lambda}) \\ \frac{3(\sin \mu + \mu \cos \mu \quad) + \sin \mu(x-1) - 2\mu \text{ co} \quad (x-1)}{(1+2\mu^2)\sin \mu + 3\mu \cos h}, \\ \qquad \lambda < 0 (\mu = 2\sqrt{-\lambda}) \end{cases}$$

6. $\varphi(x) = -1$

7. $\varphi(x) = \frac{e . \sinh\sqrt{2}x}{\sinh\sqrt{2} + \sqrt{2}\cosh\sqrt{2}}$

8. $\varphi(x) =$

$$\begin{cases} \frac{-\sinh 1 . \cos \mu x}{\mu \sin \mu}, \lambda > 1 \;\; (\mu = \sqrt{\lambda - 1}\,) \\ \frac{-\sin \text{h}\, 1 . \cos \text{h}\, \mu\, x}{\mu \sin \text{h}\, \mu}, \lambda < 1 \;\; (\mu = \sqrt{1 - \lambda}\,) \\ \text{no solutins if } \lambda = 1 \end{cases}$$

9. $$\varphi(x) = \begin{cases} \frac{\cosh \mu(x - \frac{\pi}{2})}{\cosh\frac{\mu\pi}{2} - \frac{\mu\pi}{2}\sinh\frac{\mu\pi}{2}} & \text{if } \mu = \sqrt{2\lambda},\; \lambda > 0 \\ \frac{\cos \mu(x - \frac{\pi}{2})}{\cos\frac{\mu\pi}{2} - \frac{\mu\pi}{2}\sin\frac{\mu\pi}{2}} & \text{if } \mu = \sqrt{-2\lambda},\; \lambda < 0 \end{cases}$$

$\varphi(x) \equiv 1$ if $\lambda = 0$; μ is not a root of the equations $\cosh\frac{\mu\pi}{2} - \frac{\mu\pi}{2}\sinh\frac{\mu\pi}{2} = 0$,

$$\cos\frac{\mu\pi}{2} - \frac{\mu\pi}{2}\sin\frac{\mu\pi}{2} = 0.$$

Section-B**:**

1. $\varphi(x) = \frac{e}{e-2\lambda}x, \lambda \neq \frac{e}{2}$. No solutions for $\lambda = -\frac{e}{2}$.

2. $\varphi(x) = x + \frac{2\pi^2\lambda|x-\pi|}{1-\pi^2\lambda}, \lambda \neq \frac{1}{\pi^2}$. No solution for $\lambda = \frac{1}{\pi^2}$

3. $\varphi(x) = \frac{3x(2\lambda^2x-2\lambda^2-5\lambda-6)+(\lambda+3)^2}{(\lambda+3)^2}, \lambda \neq -3$. No solutions for $\lambda = -3$.

4. $\varphi(x) = \begin{cases} x^3 - \frac{3(4\lambda+5).x}{5(4\lambda+3)} \text{ if } \lambda \neq \frac{3}{2}, \lambda \neq -\frac{3}{4} \\ x^3 - \frac{11}{15}x + Cx^2 \text{ if } \lambda = \frac{3}{2} \end{cases}$

For $\lambda = \frac{3}{4}$ there are no solution.

5. $\varphi(x) = \begin{cases} \sin x \text{ if } \lambda \neq 1 \\ C_1\cos x + C_2\sin 2x + \sin x \text{ if } \lambda = 1 \end{cases}$

6. $\varphi(x) = -\frac{x^2}{2} + \frac{3}{2} - \tanh 1$ if $\lambda = -1$; $\varphi(x) = \left\{\frac{(\mu^2-1)\cosh\mu x}{\cosh\mu - \mu\sinh\mu\, t} + 1\right\}\frac{1}{\mu^2}$

if $\lambda = \mu^2 - 1$, where μ is not a root of the equation $\cosh\mu = \mu\sinh\mu\tanh 1$; $\varphi(x) = \frac{1}{\mu^2}\left\{\frac{(\mu^2+1)\cos\mu x}{\cos\mu - \mu\sin\mu\tanh} - 1\right\}$

if $\lambda = -(\mu^2 - 1)$, where μ is not a root of the equation $\cosh\mu + \mu\sinh\mu\tanh 1 = 0$. In the remaining cases there are no solutions.

14.7 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Oybkusgers, Moscow, 1971 by M. Krsnov, A. Kiselev and G. Makarendo.
2. Integral Equations and their Applications, John wiley & Sons, 1999, by Jerri, A.
3. Linear Integral Equation, Theory and Techniques, Academic Press, 2014 by kanwal R.P.
4. A first course in Integral Equations, 2nd edition, World Scientific Publishing Co. 2015 by Wazwaz, A.M.
5. Integral equations, Krishna Prakashan Media(P) Ltd., Meerut.

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LESSON- 15

CONSTRUCTION OF GREEN'S FUNCTION FOR ORDINARY DIFFERENTIAL EQUATIONS

OBJECTIVES:

Learn to Construct Green's Function for Ordinary Differential Equations.

STRUCTURE:

15.1 Introduction

15.2 Definition of Green's Function

15.3 Summary

15.4 Technical Terms

15.5 Self-Assessment Questions

15.6 Suggested Readings

15.1 INTRODUCTION:

Green's function provides a method to solve non-homogeneous ordinary differential equations (ODEs) by finding the solution to the equation with a delta function forcing term, which allows for solving more complex problems by superposition.

15.2 DEFINITION OF GREEN'S FUNCTION:

Consider the homogeneous differential equation of order 'n' is $L[y] \equiv P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_n(x)y = 0$ (1)

where the function $P_0(x), P_1(x), \dots, P_n(x)$are continuous on $[a, b]$,$P_0(x) \neq 0$ on $[a, b]$ and the boundary conditions are $V_i(y) = 0$

$$V_i(y) = \sum_{i=1}^{n} \alpha_i^{\,n-1} y^{n-1}(a) + \sum_{i=1}^{n} \beta_i^{\,n-1} y^{n-1}(b), \qquad (2)$$

where the linear forms $V_1, V_2, \dots, V_n$ in $y(a), y'(a) \dots y^{(n-1)}(a), y(b), \dots, y^{(n-1)}(b)$

are linearly independent.

If the homogeneous boundary value problem given by equation (1) and equation (2) has only a trivial solution $y(x) \equiv 0$.

15.2.1 Construction of Green's Function: The Green's function $G(x, \xi)$, constructed for any point ξ, $a < \xi < b$, for a boundary value problem which has the following properties:

1. In each of the intervals $[a, \xi)$ and $(\xi, b]$ the function $G(x, \xi)$ considered as a function of x, is a solution of equation (1) is $L[G] = 0$.

2. $G(x,\xi)$ is continuous in x for fixed ξ and has continuous derivatives with respect to x upto order $(n-2)$ inclusive for $a \le x \le b$.
3. $(n-1)^{th}$ derivative of $G(x,\xi)$ with respect to x at $x = \xi$ has the discontinuity of the first kind, and the jump being equal to $\frac{1}{P_0(x)}$,

$$\text{i.e.,}\qquad \frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\Big]_{x=\xi+0} - \frac{\partial^{n-1}}{\partial x^{n-1}} G(x,\xi)\Big]_{x=\xi-0} = \frac{1}{P_0(\xi)}.$$

4. $G(x,\xi)$ satisfies the boundary conditions (2): $V_i(G) = 0$, $(i = 1,2,\dots,n)$

15.2.2. Theorem: If the boundary value problem

$$L[y] \equiv P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_n(x)y = 0, \tag{1}$$

$$V_i(y) = \textstyle\sum_{i=1}^{n} \alpha_i^{\,n-1} y^{n-1}(a) + \sum_{i=1}^{n} \beta_i^{\,n-1} y^{n-1}(b) \tag{2}$$

has only the trivial solution $y(x) \equiv 0$, then the operator L has one and only one Green's function $G(x,\xi)$.

Proof:

Let $y_1(x), y_2(x), \dots, y_n(x)$ be linearly independent solutions of the equation $L[y] = 0$.

Then, by virtue of property (1), the unknown function $G(x,\xi)$ must have the following representation on the intervals $[a,\xi)$ and $(\xi,b]$:

$G(x,\xi) = a_1y_1(x) + a_2y_2(x) + \cdots + a_ny_n(x)$ for $a \le x < \xi$

and $G(x,\xi) = b_1y_1(x) + b_2y_2(x) + \cdots + b_ny_n(x)$ for $\xi \le x < b$

Here, $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are some functions of ξ. The continuity of the function $G(x,\xi)$ and of its first $n-2$ derivatives with respect to x at the point $x = \xi$ yields the relations

$$[b_1y_1(\xi) + \cdots + b_ny_n(\xi)] - [a_1y_1(\xi) + \cdots + a_ny_n(\xi)] = 0,$$
$$[b_1y_1'(\xi) + \cdots + b_ny_n'(\xi)] - [a_1y_1'(\xi) + \cdots + a_ny_n'(\xi)] = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left[b_1y_1^{(n-2)}(\xi) + \cdots + b_ny_n^{(n-2)}(\xi)\right] - \left[a_1y_1^{(n-2)}(\xi) + \cdots + a_ny_n^{(n-2)}(\xi)\right] = 0$$

and property (3) takes the form

$$\left[b_1y_1^{(n-1)}(\xi) + \cdots + b_ny_n^{(n-1)}(\xi)\right] - \left[a_1y_1^{(n-1)}(\xi) + \cdots + a_ny_n^{(n-1)}(\xi)\right] = \frac{1}{P_0(\xi)}$$

Let us put $c_i(\xi) = b_i(\xi) - a_i(\xi)$ $(i = 1,2,\dots,n)$; then we get a system of linear equations in $c_i(\xi)$:

$$\left.\begin{array}{c} c_1y_1(\xi)+c_2y_2(\xi)+\cdots+c_ny_n(\xi)=0 \\ c_1y_1'(\xi)+c_2y_2'(\xi)+\cdots+c_ny_n'(\xi)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1y_1^{(n-2)}(\xi)+c_2y_2^{(n-2)}(\xi)+\cdots+c_ny_n^{(n-2)}(\xi)=0 \\ c_1y_1^{(n-1)}(\xi)+c_2y_2^{(n-1)}(\xi)+\cdots+c_ny_n^{(n-1)}(\xi)=\frac{1}{P_0(\xi)} \end{array}\right\} \quad (3)$$

The determinant of system (3) is equal to the value of the Wronskian $W(y_1, y_2, \dots, y_n)$ at the point $x = \xi$ and is therefore different from zero. For this reason, system (3) uniquely defines the functions $c_i(\xi)$ $(i = 1,2, \dots, n)$. To determine the functions $a_i(\xi)$ and $b_i(\xi)$ let us take advantage of the boundary conditions (2). We write $V_i(y)$ in the form

$$V_i(y) = A_i(y) + B_i(y) \quad (4)$$

where

$$A_i(y) = \propto_i y(a) + \propto_i^{(1)} y'(a) + \cdots + \propto_i^{(n-1)} y^{(n-1)}(a),$$

$$B_i(y) = \beta_i y(b) + \beta_i^{(1)} y'(b) + \cdots + \beta_i^{(n-1)} y^{(n-1)}(b).$$

Then, by property (4), we get

$$V_i(G) = a_1A_i(y_1) + \cdots + a_nA_i(y_n)$$

$$+b_1B_i(y_1) + b_2B_i(y_2) + \cdots + b_nB_i(y_n) = 0 \quad (i = 1,2, \dots, n)$$

Taking into consideration that $a_i = b_i - c_i$, we will have

$$(b_1 - c_1)A_i(y_1) + (b_2 - c_2)A_i(y_2) + \cdots + (b_n - c_n)A_i(y_n)$$

$$+b_1B_i(y_1) + b_2B_i(y_2) + \cdots + b_nB_i(y_n) = 0 \quad (i = 1,2, \dots, n)$$

Whence, by virtue of (4),

$$b_1V_i(y_1) + b_2V_i(y_2) + \cdots + b_nV_i(y_n) = c_1A_i(y_1) + c_2A_i(y_2) + \cdots + c_nA_i(y_n)$$

$$(i = 1,2, \dots, n) \quad (5)$$

Note that system (5) is linear in the quantities $b_1, b_2, \dots, b_n$.

The determinant of the system is different from zero:

$$\begin{vmatrix} V_1(y_1) & V_1(y_2) & \dots & V_1(y_n) \\ V_2(y_1) & V_2(y_2) & \dots & V_2(y_n) \\ \dots & \dots & \dots & \dots \\ V_n(y_1) & V_n(y_2) & \dots & V_n(y_n) \end{vmatrix} \neq 0 \quad (6)$$

by virtue of our assumption concerning the linear independence of the forms $V_1, V_2, \dots, V_n$.

Consequently, the system of equations (5) has a unique solution in $b_1(\xi), b_2(\xi), \dots, b_n(\xi)$, and since $a_i(\xi) = b_i(\xi) - c_i(\xi)$, it follows that the quantities $a_i(\xi)$ $(i = 1,2, \dots, n)$ are defined uniquely. Thus, the existence and uniqueness of Green's function $G(x, \xi)$ have been proved and a method has been given for constructing the function.

Note 1: If the boundary value problem

$L[y] \equiv P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_n(x)y = 0,$

$V_i(y) = \sum_{i=1}^{n} {\alpha_i}^{n-1} y^{n-1}(a) + \sum_{i=1}^{n} {\beta_i}^{n-1} y^{n-1}(b)$

is self-adjoint, then Green's function is symmetric, i.e., $G(x, \xi) = G(\xi, x)$

The converse is true as well.

Note 2: If at one of the extremities of an interval $[a, b]$ the coefficient of the highest derivative vanishes, for example $p_0(a) = 0$, then the natural boundary condition for boundedness of the solution at $x = a$ is imposed, and at the other extremity, the ordinary boundary condition is specified (see example 2 below).

15.2.3 An important special case:

Let us consider the construction of Green's function for a second order differential equation of the form

$(p\,(x)y')' + q(x)\,y \;=\; 0,$

$p\,(x) \;\neq 0$ on $[a, b]$, $p(x) \in C^{(1)}[a, b]$ (1)

with boundary conditions

$$y(a) = \; y\,(b) = 0 \quad (2)$$

Suppose that $y_1(x)$ is a solution of equation (1) defined by the initial conditions

$y_1(a)$=0, $y_1'(a)$=$\alpha \neq 0$ (3)

Generally speaking, this solution need not necessarily satisfy the second boundary condition; we will therefore assume that $y_1(b)$ ≠0. But functions of the form $C_1y_1(x)$, where C_1 is an arbitrary constant, are obviously solutions of equation (1) and satisfy the boundary condition

$$y(a) \;=\; 0$$

Similarly, we find the nonzero solution $y_2(x)$ of equation (1), such that it should satisfy the second boundary condition, i.e.,

$$y_2(b) = 0 \quad (4)$$

This same condition will be satisfied by all solutions of the family $C_2y_2(x)$, where $C_2$is an arbitrary constant.

We now seek Green's function for the problem (1) - (2) in the form,

$$G(x, \xi) = \begin{cases} C_1y_1(x) & \text{for} \quad a \leq x \leq \xi \\ C_2y_2(x) & \text{for} \quad \xi \leq x \leq b \end{cases} \quad (5)$$

and we shall choose the constants C_1 and C_2 so that the properties (2) and (3) are fulfilled, i.e., so that the function $G(x, \xi)$ is continuous in x for fixed ξ, in particular, continuous at the point $x = \; \xi$;

$$C_1y_1(\xi) = C_2y_2(\xi)$$

and so that $G_x'(x,\xi)$ has a jump, at the point $x = \xi$, equal to $\frac{1}{p(\xi)}$:

$$C_2 y_2'(\xi) - C_1 y_1'(\xi) = \frac{1}{p(\xi)}$$

Rewrite the last two equalities as

$$\left.\begin{aligned} -C_1 y_1(\xi) + C_2 y_2(\xi) &= 0 \\ -C_1 y_1'(\xi) + C_2 y_2'(\xi) &= \frac{1}{p(\xi)} \end{aligned}\right\} \qquad (6)$$

The determinant of system (6) is the Wronskian $W[y_1(x), y_2(x)] = W(x)$ computed at the point $x = \xi$ for linearly independent solutions $y_1(x)$ *and* $y_2(x)$of equation (1), and, hence, it is different from zero:

$$W(\xi) \neq 0$$

So that the quantities C_1 and C_2 of the system (6) are determined at once:

$$C_1 = \frac{y_2(\xi)}{p(\xi)W(\xi)}, C_2 = \frac{y_1(\xi)}{p(\xi)W(\xi)} \qquad (7)$$

Substituting the expressions for C_1 and C_2into (5), we finally get

$$G(x,\xi) = \begin{cases} \frac{y_1(x).y_2(\xi)}{p(\xi)W(\xi)} \text{ for } a \leq x \leq \xi \\ \frac{y_1(\xi).y_2(x)}{p(\xi)W(\xi)} \text{ for } \xi \leq x \leq b \end{cases} \qquad (8)$$

Note 1. The solution $y_1(x)$ and $y_2(x)$ of equation (1) that we have chosen are linearly independent by virtue of the assumption that $y_1(b) \neq 0$.

Indeed, all solutions are linearly dependent on $y_1(x)$ have the form $C_1 y_1(x)$ and, consequently, for $C_1 \neq 0$, do not vanish at the point $x = b$ at which, according to our choice, the solution $y_2(x)$ vanishes.

Note 2. The boundary value problem for a second order equation of the form

$$y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0 \qquad (9)$$

and boundary conditions $y(a) = A, y(b) = B$ (10)

reduces to the above considered problem (1)-(2) as follows:

(i) The linear equation (9) is reduced to (1) by multiplying (9) by $p(x) = e^{\int p_1(x)dx}$ [we have to take $p(x)p_2(x)$ *for* $q(x)$].

(ii) The boundary conditions (10) reduced to zero conditions (2) by linear change of variables

$$z(x) = y(x) - \frac{B-A}{b-a}(x-a) - A$$

The linearity of equation (9) is preserved in this change, but unlike equation (1), we now obtain the non-homogeneous equation $L(z) = f(x)$, where

$$f(x) = -\left[A + \frac{B-A}{b-a}(x-a)\right]q(x) - \frac{B-A}{b-a}p(x)$$

However, we construct Green's function for the homogeneous boundary value problem $L[z]=0, z(a)=z(b)=0$, which entirely coincides with the problem (1) – (2).

15.2.3.1 Example 1: Construct Green's function for the homogeneous boundary value problem

$$y^{IV}(x)=0 \qquad (1)$$

$$\left.\begin{matrix} y(0)=y'(0)=0 \\ y(1)=y'(1)=0 \end{matrix}\right\} \qquad (2)$$

Solution: We shall first show that the boundary value problem (1) - (2) has only a trivial solution. Indeed, the fundamental system of solutions for equation (1) is

$$y_1(x)=1, y_2(x)=x, y_3(x)=x^2, y_4(x)=x^3, \qquad (3)$$

so that general solution is of the form

$$y(x)=A+Bx+Cx^2+Dx^3,$$

where A, B, C, D are arbitrary constants. The boundary conditions (2) give us four relations for determining A, B, C, D:

$$y(0)=A=0,$$

$$y'(0)=B=0,$$

$$y(1)=A+B+C+D=0$$

$$y'(1)=B+2C+3D=0$$

Whence, we have $A=B=C=D=0$

Thus, the problem (1) - (2) has only a zero solution $y(x)\equiv 0$, and, hence, for it we can construct a (unique) Green's function $G(x,\xi)$.

We now construct the Green's function. Utilizing the fundamental system of solutions (3), represent the unknown Green's function $G(x,\xi)$ in the form

$$G(x,\xi)= a_1.1+a_2.x+a_3.x^2+a_4.x^3 \text{ for } 0\le x\le \xi \qquad (4)$$

$$G(x,\xi)= b_1.1+b_2.x+b_3.x^2+b_4.x^3 \text{ for } \xi\le x\le 1 \qquad (5)$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are as yet unknown functions of ξ. Put $c_i(\xi)=b_i(\xi)-a_i(\xi)$ $(i=1,2,3,4)$ and write out the system of linear equations for finding the functions $c_i(\xi)$

see the system

$$\left.\begin{matrix} c_1y_1(\xi)+c_2y_2(\xi)+\cdots+c_ny_n(\xi)=0 \\ c_1y_1'(\xi)+c_2y_2'(\xi)+\cdots+c_ny_n'(\xi)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1y_1^{(n-2)}(\xi)+c_2y_2^{(n-2)}(\xi)+\cdots+c_ny_n^{(n-2)}(\xi)=0 \\ c_1y_1^{(n-1)}(\xi)+c_2y_2^{(n-1)}(\xi)+\cdots+c_ny_n^{(n-1)}(\xi)=\frac{1}{P_0(\xi)} \end{matrix}\right\}$$

Therefore, we get

$$\left.\begin{aligned} c_1 + c_2\xi + c_3\xi^2 + c_4\xi^3 = 0, \\ c_2 + 2c_3\xi + 3c_4\xi^2 = 0, \\ 2c_3 + 6c_4\xi = 0, \\ 6c_4 = 1. \end{aligned}\right\} \quad (6)$$

Solving the system, we get

$$\left.\begin{aligned} c_1(\xi) = -\frac{1}{6}\xi^3, c_2(\xi) = \frac{1}{2}\xi^2 \\ c_3(\xi) = -\frac{1}{2}\xi, c_4(\xi) = \frac{1}{6} \end{aligned}\right\} \quad (7)$$

We further take advantage of property (4) of Green's function, namely, that it must satisfy the boundary conditions (2), i.e.,

$$G(0,\xi) = 0, G_x'(0,\xi) = 0$$

$$G(1,\xi) = 0, G_x'(1,\xi) = 0$$

In our case, these relations take the form

$$\left.\begin{aligned} a_1 = 0 \\ a_2 = 0 \\ b_1 + b_2 + b_3 + b_4 = 0 \\ b_2 + 2b_3 + 3b_4 = 0 \end{aligned}\right\} \quad (8)$$

Taking advantage of the fact that $c_k = b_k - a_k$(k=1, 2, 3, 4), we find from (7) and (8) that

$$\left.\begin{aligned} a_1 = 0;\ a_2 = 0;\ b_1 = -\frac{1}{6}\xi^3; b_2 = \frac{1}{2}\xi^2; \\ b_3 = \frac{1}{2}\xi^3 - \xi^2;\ b_4 = \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3; \\ a_3 = \frac{1}{2}\xi - \xi^2 + \frac{1}{2}\xi^3;\ a_4 = -\frac{1}{6} + \frac{1}{2}\xi^2 - \frac{1}{3}\xi^3 \end{aligned}\right\} \quad (9)$$

Putting the values of the coefficients $a_1, a_2, \dots, b_4$ from (9) into (4) and (5), we obtain the desired Green's function:

$$G(x,\xi) = \begin{cases} \left(\frac{1}{2}\xi - \xi^2 + \frac{1}{2}\xi^3\right)x^2 - \left(\frac{1}{6} - \frac{1}{2}\xi^2 + \frac{1}{3}\xi^3\right)x^3, \\ \qquad 0 \le x \le \xi \\ -\frac{1}{6}\xi^3 + \frac{1}{2}\xi^2 x + \left(\frac{1}{2}\xi^3 - \xi^2\right)x^2 + \left(\frac{1}{2}\xi^2 - \frac{1}{3}\xi^3\right)x^3, \\ \qquad \xi \le x \le 1 \end{cases}$$

This expression is readily transformed to

$$G(x,\xi) = \left(\frac{1}{2}x - x^2 + \frac{1}{2}x^3\right)\xi^2 - \left(\frac{1}{6} - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\xi^3 \quad \text{for} \quad \xi \le x \le 1$$

So that $G(x,\xi) = G(\xi,x)$, i.e., Green's function is symmentric. This was evident from the start since the boundary- value problem (1)-(2) was self-adjoint.

15.2.3.2 Example 2. Construct Green's function for the differential equation

$$xy'' + y' = 0 \quad (1)$$

For the following conditions:

$y(x)$ bounded as $x \to 0$,

$y(1) = \alpha y'(1), \alpha \neq 0$ (2)

Solution. First find the general solution of equation (1) and convince yourself that the conditions (2) are fulfilled only when $y(x) \equiv 0$

Indeed, denoting $y'(x) = z(x)$ we get $xz' + z = 0$, whence $\ln z = \ln c_1 - \ln x, z = \frac{c_1}{x}$ and, hence,

$$y(x) = c_1 \ln x + c_2 \quad (3)$$

It is clear that $y(x)$ defined by formula (3) satisfies the conditions (2) only for $c_1 = c_2 = 0$, and, hence, Green's function can be constructed for the problem (1)-(2).

Let us write down $G(x, \xi)$ formally as

$$G(x,\xi) = \begin{cases} a_1 + a_2 \ln x & \text{for } 0 < x \leq \xi, \\ b_1 + b_2 \ln x & \text{for } \xi \leq x \leq 1 \end{cases} \quad (4)$$

From the continuity of $G(x, \xi)$ for $x = \xi$ we obtain

$$b_1 + b_2 \ln \xi - a_1 - a_2 \ln \xi = 0$$

and the jump $G'_x(x, \xi)$ at the point $x = \xi$ is equal to $\frac{1}{\xi}$ so that

$$b_2.\frac{1}{\xi} - a_2.\frac{1}{\xi} = \frac{1}{\xi}$$

Putting $c_1 = b_1 - a_1, \;\; c_2 = b_2 - a_2$ (5)

we will have

$$\begin{cases} c_1 + c_2 \ln \xi = 0, \\ \qquad\quad c_2 = 1 \end{cases}$$

whence

$$c_1 = -\ln \xi, \quad c_2 = 1 \quad (6)$$

Now let us use conditions (2). The boundedness of $G(x, \xi)$ as $x \to 0$ gives us $a_2 = 0$, and from the condition $G(x, \xi) == \alpha G'_x(x, \xi)$ we get $b_1 = \alpha b_2$. Taking into account (5) and (6), we get the values of all coefficients in (4):

$$a_1 = \alpha + \ln \xi, \;\; a_2 = 0, \;\; b_1 = \alpha, \;\; b_2 = 1$$

Thus

$$G(x,\xi) = \begin{cases} \alpha + \ln \xi, & 0 < x \leq \xi, \\ \alpha + \ln x, & \xi \leq x \leq 1 \end{cases}$$

15.2.3.3 Example 3. Find Green's function for the boundary-value problem

$$y''(x) + k^2 y = 0$$
$$y(0) = y(1) = 0$$

Solution. It is easy to see that the solution $y_1(x) = \sin kx$ satisfies the boundary condition $y_1(0) = 0$, and the solution $y_2(x) = \sin k(x-1)$ satisfies the condition $y_2(1) = 0$; they are linearly independent. Let us find the value of the Wronskian for $\sin kx$ and $\sin k(x-1)$ at the point $x = \xi$:

$$W(\xi) = \begin{vmatrix} \sin k\xi & \sin k(\xi - 1) \\ k\cos k\xi & k\cos k(\xi - 1 \end{vmatrix}$$
$$= k[\sin k\xi \cos k(\xi - 1) - \sin k\,(\xi - 1)\cos k\xi] = k \sin k$$

Noting, in addition, that in our example $p(x) = 1$, we get, by (8) from 15.2.3,

$$G(x,\xi) = \begin{cases} \dfrac{\sin k(\xi - 1)\sin kx}{k\sin k}, & 0 \le x \le \xi, \\ \dfrac{\sin k\xi \sin k\,(x-1)}{k \sin k}, & \xi \le x \le 1 \end{cases}$$

15.2.3.3 Example 4: Find the Green's function for the boundary value problem

$$y'' = 0; y(0) = y'(1), y'(0) = y(1).$$

Solution:

Given homogeneous differential equation is $y'' = 0$ (1)

The general solution of the differential equation (1) is

$$y(x) = A + Bx, A \,\&\, B \text{ are arbitrary constants.} \qquad (2)$$

Applying the given boundary conditions in equation (2), we get $A = 0$ and $B = 0$,

Thus , the solution of equation (1) is $y(x) = 0$.

Construction of Green's function: $G(x,\xi)$

$$G(x,\xi) = \begin{cases} a_1 + a_2 x, 0 \le x \le \xi \\ b_1 + b_2 x, \xi \le x \le 1, \end{cases}$$

where a_1, a_2, b_1, b_2 are functions of ξ.

Since $G(x,\xi)$ must be continuous at $x = \xi$, we get

$$a_1 + a_2\xi = b_1 + b_2\xi \qquad (3)$$

and the jump condition for $G_x(x,\xi)$ ($\because$ from the definition of Green's function, the derivative has a unique jump at $x = \xi$) gives

$$b_2 - a_2 = 1 \qquad (4)$$

using the boundary conditions

$G(0,\xi) = G_x(1,\xi) \Rightarrow a_1 = b_2$

$G_x(0,\xi) = G(1,\xi) \Rightarrow a_2 = b_1 + b_2$ (5)

Solve the coefficients a_1, a_2, b_1, b_2, from equations (3), (4), & (5). i.e.,

substitute $a_1 = b_2$ into the continuity equation $b_2 + a_2\xi = b_1 + b_2\xi$,

using jump condition $b_2 = a_2 + 1 \Rightarrow a_2 + 1 = b_1 + \xi$

using $a_2 = b_1 + b_2 \Rightarrow b_2 = \xi - 1$.

Since $a_1 = b_2, a_1 = \xi - 1$

using $b_2 - a_2 = 1$, we get $a_2 = \xi - 2$.

using $a_2 = b_1 + b_2 \Rightarrow b_1 = -1$

$$\therefore\ G(x,\xi) = \begin{cases} (\xi - 1) + (\xi - 2)x, & 0 \le x \le \xi \\ -1 + (\xi - 1)x, & \xi \le x \le 1 \end{cases}.$$

15.3 SUMMARY:

This chapter(lesson) provided the definition of Green's function and its construction. It is observed that the Green's function provides a method to solve homogeneous ODE's by finding the solution to the equation with a delta function forcing terms. A theorem related to the solution of the BVP and green's function is stated along with its proof. The procedure is explained well with the help of few examples and self-assessment problems were given at the end.

15.4 TECHNICAL TERMS:

Green's Function: Consider the homogeneous differential equation of order 'n' is $L[y] \equiv P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_n(x)y = 0$ (1)

where the function $P_0(x), P_1(x), \dots, P_n(x)$ are continuous on $[a,b]$,$P_0(x) \neq 0$ on $[a,b]$ and the boundary conditions are $V_i(y) = 0$

$V_i(y) = \sum_{i=1}^{n} {\alpha_i}^{n-1} y^{n-1}(a) + \sum_{i=1}^{n} {\beta_i}^{n-1} y^{n-1}(b)$, (2)

where the linear forms $V_1, V_2, \dots, V_n$ in $y(a), y^1(a) \dots y^{n-1}(a), y(b), \dots, y^{n-1}(b)$

are linearly independent.

If the homogeneous boundary value problem given by equation (1) and equation (2) has only a trivial solution $y(x) \equiv 0$.

15.5 SELF-ASSESSMENT QUESTIONS:

Construct Green's function for the following boundary value problem

1. $y'' = 0; y(0) = y(1),\ y'(0) = y'(1)$.

2. $y'' + y = 0; y(0) = y(\pi) = 0$.

3. $y^{\mathrm{IV}} = 0; y(0) = y'(0) = y''(1) = y'''(1) = 0.$

4. $y''' = 0; y(0) = y'(1) = 0; y'(0) = y(1).$

5. $y''' = 0; y(0) = y(1) = 0; y'(0) = y'(1).$

6. $y'' = 0; y(0) = 0,\ y(1) = y'(1).$

7. $y'' + y' = 0; y(0) = y(1), y'(0) = y'(1).$

8. $y'' - k^2 y = 0 (k \neq 0); y(0) = y(1) = 0.$

9. $y'' + y = 0; y(0) = y(1), y'(0) = y'(1).$

10. $y''' = 0; y(0) = y(1) = 0, y'(0) + y'(1) = 0.$

11. $y'' = 0; y'(0) = hy(0), y'(1) = -Hy(1).$

12. $x^2 y'' + 2xy' = 0; y(x)$ is bounded for $x \to 0, y(1) = \alpha y'(1).$

13. $x^3 y^{IV} + 6x^2 y''' + 6xy'' = 0; y(x)$ is bounded as $x \to 0, y(1) = y'(1) = 0$

14. $y'' + xy' - y = 0; y(x)$ is bounded for $x \to 0, y(1) = 0$

15. $xy'' + y' - \frac{1}{x} y = 0; y(0)$ is finite, $y(1) = 0$

16. $x^2 y'' + xy' - n^2 y = 0;\ y(0)$ is finite, $y(1) = 0$

17. $x^2(ln\, x - 1)y'' - xy' + y = 0;\ y(0)$ is finite, $y(1) = 0$

18. $\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] = 0; y(0) = 0, y(1)$ is finite..

19. $xy^n + y^1 = 0; y(0)$ is bounded, $y(l) = 0,$

20. $y^n - y = 0; y(0) = y^1(0), y(l) + \lambda y'(l) = 0.$(consider the cases: λ =1, λ =-1, | λ | $\neq$ 1)

Answers:

1. It is obvious that the equation $y''(x) = 0$ has an infinity of solutions $y(x) = C$ under the conditions $y(0) = y(1), y'(0) = y'(1)$. Therefore, Green's function does not exist for this boundary-value problem.

2. Green's function does not exist.

3. $G(x,\xi) = \begin{cases} \frac{x^2}{6}(3\xi - x), 0 \le x \le \xi \\ \frac{\xi^2}{6}(3x - \xi), \xi \le x \le 1 \end{cases}$

4. $G(x,\xi) = \begin{cases} \frac{x(\xi-1)}{2}(x - x\xi + 2\xi), \quad 0 \le x \le \xi \\ \frac{\xi}{2}[x(2 - x)(\xi - 2) + \xi],\ \xi \le x \le 1 \end{cases}$

5. $G(x,\xi)=\begin{cases}\frac{x(x-\xi)(\xi-1)}{2}, & 0\le x\le \xi\\ \frac{-\xi(\xi-x)(x-1)}{2}, & \xi\le x\le 1\end{cases}$

6. Green's function does not exist.

7. Green's function does not exist

8. $G(x,\xi)=\begin{cases}\frac{\sinh k(\xi-1)\,\sinh kx}{k\,\text{si}}, & 0\le x\le \xi\\ \frac{\sin k\xi\,\sinh\,k(x-1)}{k\sinh}, & \xi\le x\le 1\end{cases}$

9. $G(x,\xi)=\begin{cases}\frac{\cos\,(x-\xi+\frac{1}{2})}{2\sin\frac{1}{2}}, & 0\le x\le \xi\\ \frac{\cos\,(\xi-x+\frac{1}{2})}{2\sin\frac{1}{2}}, & \xi\le x\le 1\end{cases}$

10. Green's function does not exist.

11. $G(x,\xi)=\begin{cases}\frac{(hx+1)[H(\xi-1)-1]}{h+H+hH}, & 0\le x\le \xi\\ \frac{(h\xi+1)[H(x-1)-1]}{h+H+hH}, & \xi\le x\le 1\end{cases}$

12. $G(x,\xi)=\begin{cases}\alpha+1-\frac{1}{\xi}, & 0\le x\le \xi\\ \alpha+1-\frac{1}{x}, & \xi\le x\le 1\end{cases}$

13. $G(x,\xi)=\begin{cases}\xi-\ln\xi-1-\frac{x(\xi-1)^2}{2\xi}, & 0\le x\le \xi\\ x-\ln x-1-\frac{\xi(x-1)^2}{2x}, & \xi\le x\le 1\end{cases}$

14. $G(x,\xi)=\begin{cases}\frac{x}{2}\left(1-\frac{1}{\xi^2}\right), & 0\le x\le \xi\\ \frac{1}{2}\left(x-\frac{1}{x}\right), & \xi\le x\le 1\end{cases}$

15. $G(x,\xi)=\begin{cases}\frac{x}{2}\left(\xi-\frac{1}{\xi}\right), & 0\le x\le \xi\\ \frac{\xi}{2}\left(x-\frac{1}{x}\right), & \xi\le x\le 1\end{cases}$

16. $G(x,\xi)=\begin{cases}\frac{1}{2n\xi}[(x\xi)^n-\left(\frac{x}{\xi}\right)^n, & 0\le x\le \xi\\ \frac{1}{2n\xi}[(x\xi)^n-\left(\frac{\xi}{x}\right)^n, & \xi\le x\le 1\end{cases}$

17. $G(x,\xi) = \begin{cases} -\frac{x\ln}{\xi^2(\ln\xi-1)^2}, 0 \le x \le \xi \\ -\frac{\ln x}{\xi(\ln\xi-1)^2}, \xi \le x \le 1 \end{cases}$

18. $G(x,\xi) = \begin{cases} \frac{1}{2}\ln\frac{1-x}{1+x}, 0 \le x \le \xi \\ \frac{1}{2}\ln\frac{1-\xi}{1+\xi}, \xi \le x \le 1 \end{cases}$

19. $G(x,\xi) = \begin{cases} \ln\frac{\xi}{l}, 0 \le x \le \xi \\ \frac{1}{2}\ln\frac{x}{l}, \xi \le x \le l \end{cases}$

20. $G(x,\xi) =$

$$\begin{cases} \left[\frac{1-\lambda}{2(1+\lambda)}e^{\xi-2l} - \frac{1}{2}e^{-\xi}\right]e^{x}, 0 \le x \le \xi \quad (|\lambda| \ne 1) \\ \frac{1-\lambda}{2(1+\lambda)}e^{\xi-2l} - \frac{1}{2}e^{\xi-x}, \qquad \xi \le x \le l \end{cases}$$

For $\lambda = 1$, $G(x,\xi) = -\frac{1}{2}e^{-|x-\xi|}$ does not depend on *l.*

For $\lambda = -1$, Green's function does not exist

15.6 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Oybkusgers, Moscow, 1971 by M. Krsnov, A. Kiselev and G. Makarendo.
2. Integral Equations and their Applications, John wiley & Sons, 1999, by Jerri, A.
3. Linear Integral Equation, Theory and Techniques, Academic Press, 2014 by kanwal R.P.
4. A first course in Integral Equations, 2nd edition, World Scientific Publishing Co. 2015 by Wazwaz, A.M.
5. Integral equations, Krishna Prakashan Media(P) Ltd., Meerut.

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- **Dr. Vinutha Tummala**

LESSON- 16

USING GREEN’S FUNCTION IN THE SOLUTION OF BOUNDARY-VALUE PROBLEMS

OBJECTIVES:

Learn to Solve Boundary Value Problems by Using Green’s Function.

STRUCTURE:

16.1 Introduction

16.2 Summary

16.3 Technical Terms

16.4 Self-Assessment Questions

16.5 Suggested Readings

16.1 INTRODUCTION:

Let there be given a non-homogeneous differential equation

$$L[y] = p_0(x)y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = f(x) \qquad (1)$$

and the boundary conditions

$$V_1(y) = 0, V_2(y) = 0, \dots, V_n(y) = 0, \qquad (2)$$

As in lesson (15), we consider that the linear form $V_1, V_2, \dots, V_n$ in $y(a), y'(a) \dots y^{(n-1)}(a), y(b), y'(b) \dots, y^{(n-1)}(b)$ are linear independent.

16.1.1 Theorem:

If $G(x, \xi)$ Green’s function of the homogeneous boundary value problem $L[y]$ =0, $V_k(y) = 0, (k = 1,2,3, \dots, n)$.

then the solution of the boundary value problem (1)-(2) is given by the formula

$$y(x) = \int_a^b G(x, \xi)\, f(\xi)\, d\xi \qquad (3)$$

16.1.2 Example 1:

Using Green’s function, solve the boundary value problem

$$y''(x) - y(x) = x, \qquad (1)$$

$$y(0) = y(1) = 0 \qquad (2)$$

Solution: a) Let us first find out whether Green's function exists for the corresponding homogeneous boundary value problem

$y''(x) - y(x) = 0,$ (1′)

$y(0) = y(1) = 0$ (2′)

It is obvious that $y_1(x) = e^x, y_2(x) = e^{-x}$ is the fundamental system of solutions of the equation (1′). Hence, the general solution of the equation is

$$y(x) = Ae^x + Be^{-x}$$

The boundary conditions (2) are satisfied if and only if $A = B = 0$, i.e., $y(x) \equiv 0$. Thus, Green's function exists.

b) It can readily be verified that

$$G(x,\xi) = \begin{cases} \frac{\sinh x \sinh(\xi-1)}{\sinh}, & 0 \le x \le \xi \\ \frac{\sinh \xi \sinh(x-1)}{\sinh}, & \xi \le x \le 1 \end{cases} \quad (3)$$

is Green's function for the boundary value problem (1′) - (2′).

c) We write the solution of the boundary value problem (1) - (2) in the form

$$y(x) = \int_0^1 G(x,\xi)\, \xi \, d\xi \,, \quad (4)$$

where $G(x,\xi)$is defined by formula (3).

Splitting up the interval of integration into two parts and substituting from (3) into (4) the expression for Green's function, we obtain

$$y(x) = \int_0^x \frac{\sinh \xi \sinh(x-1)}{\sinh 1} \xi \, d\xi + \int_x^1 \frac{\sinh x \sinh(\xi - 1)}{\sinh 1} \xi \, d\xi$$

$$y(x) = \frac{\sinh(x-1)}{\sinh} \int_0^x \xi \sinh \xi \, d\xi + \frac{\sinh}{\text{si}} \int_x^1 \xi \sinh(\xi - 1)\, d\xi \quad (5)$$

But $\int_0^x \xi \sinh \xi \, d\xi = x \cosh x - \sinh x,$

$$\int_x^1 \xi \sinh(\xi - 1)\, d\xi = 1 - x\cosh(x-1) + \sinh(x-1)$$

and therefore

$$y(x) = \frac{1}{\sin h1}\{sin\, h(x-1)\,[x \cos hx - \sin hx] +$$

$$\sinh x\,[1 - x\cosh(x-1) + \sinh(x-1)]\} = \frac{\sinh x}{\sinh 1} - x$$

Here, we take advantage of the formula

$$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh\beta$$

and also the oddness of the function $\sinh x$.

Direct verification convinces us that the function

$$y(x) = \frac{\sinh x}{\sinh 1} - x$$

satisfies equation (1) and the boundary conditions (2).

16.1.3 Example 2:

Reduce to an integral equation the following boundary-value problem for the nonlinear differential equation:

$$y''(x) = f\big(x, y(x)\big), \tag{1}$$

$$y(0) = y(1) = 0 \tag{2}$$

Solution:

Constructing Green's function for the problem

$$y''(x) = 0, \tag{3}$$

$$y(0) = y(1) = 0 \tag{2}$$

the general solution of the equation (3) is

$$y(x) = A + Bx$$

and we find Green's function is

$$G(x,\xi) = \begin{cases} (\xi-1)x, & 0 \le x \le \xi \\ (x-1)\xi, & \xi \le x \le 1 \end{cases}$$

Regarding the right side of equation (1) as the known function, we get

$$y(x) = \int_0^1 G(x,\xi) f\big(\xi, y(\xi)\big)\, d\xi \tag{4}$$

Thus, the solution of the boundary value problem (1) – (2) reduces to the solution of a nonlinear integral equation of the Hammerstein type (see section), the kernel of which is Green's function for the problem (3)-(2). The significance of the Hammerstein- type integral equations lies precisely in the fact that the solution of many boundary value problems for nonlinear differential equations reduces to the solution of integral equations of this type.

16.1.4 Example 3:

Solve the boundary value problem using Green's function $y'' + y = x,\ y(0) = y\left(\frac{\pi}{2}\right) = 0$

Solution: The boundary value problem is

$$y'' + y = x, \qquad (1)$$

$$y(0) = y\left(\frac{\pi}{2}\right) = 0 \qquad (2)$$

Consider the boundary value problem is

$$y'' + y = 0, \qquad (3)$$

$$y(0) = y\left(\frac{\pi}{2}\right) = 0 \qquad (2)$$

The general solution of equation (3) is

$$y(x) = Acosx + Bsinx$$

and we find Green's function as

$$G(x,\xi) = \begin{cases} A_1 \cos x + B_1 \sin x\,, & for\ 0 \le x \le \xi \\ A_2 \cos x + B_2 \sin x\,, & for\ \xi \le x \le \frac{\pi}{2} \end{cases} \qquad (4)$$

Applying boundary conditions,

we obtain $G(0,\xi) = G\left(\frac{\pi}{2},\xi\right) = 0$

thus,

$$G(x,\xi) = \begin{cases} B_1 \sin x\,, & for\ 0 \le x \le \xi \\ A_2 \cos x\,, & for\ \xi \le x \le \frac{\pi}{2}, \end{cases}$$

since $G(x,\xi)$ is continuous at $x = \xi$.

16.2 SUMMARY:

This lesson is the extension of the topics discussed in the previous lesson. This lesson provides a method to solve Boundary value problem by using Green's function. Few examples were given to illustrate the method to convert a given BVP to its equivalent integral equation and there by its solution is obtained by using Green's function. To help the reader in better understanding of the discussed topics self-assessment problems were provided at the end and their answers for checking.

16.3 TECHNICAL TERMS:

GREEN'S FUNCTION: Consider the homogeneous differential equation of order 'n' is

$$L[y] \equiv P_0(x)y^n + P_1(x)y^{n-1} + \cdots + P_n(x)y = 0 \qquad (1)$$

where the function $P_0(x), P_1(x), \dots, P_n(x)$ are continuous on $[a,b]$,$P_0(x) \neq 0$ on $[a,b]$ and the boundary conditions are $V_i(y) = 0$

$$V_i(y) = \sum_{i=1}^{n} \alpha_i^{\,n-1} y^{n-1}(a) + \sum_{i=1}^{n} \beta_i^{\,n-1} y^{n-1}(b), \qquad (2)$$

where the linear forms $V_1, V_2, \dots, V_n$ in $y(a), y^1(a) \dots y^{n-1}(a), y(b), \dots, y^{n-1}(b)$

are linearly independent.

If the homogeneous boundary value problem given by equation (1) and equation (2) has only a trivial solution $y(x) \equiv 0$.

16.4 SELF-ASSESSMENT QUESTIONS:

Solve the following boundary value problems using Green's function:

1. $y^{IV} = 1; y(0) = y^{'(0)} = y''(1) = y^{'''}(1) = 0.$

2. $xy'' + y' = x; y(1) = y(e) = 0.$

3. $y'' + \pi^2 y = \cos \pi x \ ; y(0) = y(1), y'(0) = y'(1).$

4. $y'' - y = 2 \sinh 1 ; \ y(0) = \ y(1) = 0,$

5. $y'' - \ y = -2e^x; y(0) = y' \ (0), y(l) + y' \ (l) = 0.$

6. $y'' + y = x^2; y(0) = y\left(\frac{\pi}{2}\right) = 0.$

Answers:

1. $y = \frac{x^2}{24}(x^2 - 4x + 6)$

2. $y = \frac{1}{4}[(1 - e^2) \ln x + x^2 - 1]$

3. $y = \frac{1}{4\pi}[(2x - 1) \sin \pi x$

4. $y = 2[\sinh x - \sinh(x - 1) - \sinh 1]$

5. $y = \sinh x + (l - x)e^x$

6. $y = 2 \cos x + \left(2 - \frac{\pi^2}{4}\right) \sin x + x^2 - 2.$

16.5 SUGGESTED READINGS:

1. Problems and Exercises in Integral Equations, MIR Oybkusgers, Moscow, 1971 by M. Krsnov, A. Kiselev and G. Makarendo.
2. Integral Equations and their Applications, John wiley & Sons, 1999, by Jerri, A.
3. Linear Integral Equation, Theory and Techniques, Academic Press, 2014 by kanwal R.P.
4. A first course in Integral Equations, 2nd edition, World Scientific Publishing Co. 2015 by Wazwaz, A.M.
5. Integral equations, Krishna Prakashan Media(P) Ltd., Meerut.

- **Dr. Vinutha Tummala**

LESSON- 17

BOUNDARY VALUE PROBLEMS CONTAINING A PARAMETER: REDUCING THEM TO INTEGRAL EQUATIONS

OBJECTIVES:

- To learn about BVPs containing a Parameter.
- Reducing BVPs with parameters to Integral Equations.

STRUCTURE:

17.1 Introduction

17.2 Boundary Value Problems containing a parameter:

Reducing them to Integral Equations

17.3 Summary

17.4 Technical Terms

17.5 Self-Assessment Questions

17.6 Suggested Readings

17.1 INTRODUCTION:

Integral equations form one of the most useful techniques in many branches of pure analysis, such as the theory of functional analysis and stochastic processes.

It is one of the most important branches of mathematical analysis, for its importance in BVPs in ODEs and PDEs. They occur in many fields of mechanics and mathematical physics. Integral equations come from many physical problems, such as the radiation transfer problem and the neutron diffusion problem. They also come as a representation formula for the solution of differential equations. The differential equation can be replaced by an integral equation with the help of initial and boundary conditions. Each solution to the integral equation automatically satisfies the BCs.

17.1.1 Integral Equation: An integral equation is an equation in which an unknown function, to be determined, appears under one or more integral signs.

An integral equation is called linear if only linear operations are performed in it upon the known functions. For Example,

$$y(x) = f(x) + \lambda \int_a^b K(x,\xi) y(\xi)\, d\xi \qquad (1)$$

$$y(x) = \lambda \int_a^b K(x,\xi) y(\xi)\, d\xi \qquad (2)$$

17.1.2 Singular Integral Equation: When one or both limits of Integration become infinite, or the kernel becomes infinite at one or more points within the range of integration, the integral equation is called a singular integral equation. For example,

$$y(x) = f(x) + \lambda \int_{-\infty}^{\infty} e^{-|x-\xi|}\, y(\xi)\; d\xi$$

and $f(x) = \int_a^x (x-\xi)^{-\alpha}\, y(\xi)\, d\xi, \qquad 0 < \alpha < 1.$

are called singular integral equations. The second equation represents Abel's Integral equation for $\alpha = -1/2$.

17.2 BOUNDARY VALUE PROBLEMS CONTAINING A PARAMETER: REDUCING THEM TO INTEGRAL EQUATIONS:

Many situations require the consideration of a boundary-value problem of the following type:

$$L[y] = \lambda y + h(x), \qquad (1)$$

$$V_k(y) = 0 \quad (k = 1, 2, \dots, n) \qquad (2)$$

where

$$L(y) \equiv p_0(x) y^{(n)}(x) + p_1(x) y^{(n-1)}(x) + \cdots + p_n(x) y(x)$$

$$V_k(y) \equiv \alpha_k\, y(a) + \alpha_k^{(1)}\, y'(a) + \cdots + \alpha_k^{(n-1)}\, y^{(n-1)} a +$$
$$\beta_k\, y(b) + \beta_k^{(1)}\, y'(b) + \cdots + \beta_k^{(n-1)}\, y^{(n-1)}(b) \; (k = 1, 2, \dots, n)$$

(the linear forms $V_1, V_2, \dots, V_n$ are linearly independent); $h(x)$ is a given continuous function of x; λ is some numerical parameter. For $h(x) \equiv 0$, we have the homogeneous boundary–value problem

$$\left.\begin{array}{l} L[y] = \lambda y, \\ V_k(y) = 0 \;\; (k = 1, 2, \dots, n) \end{array}\right\} \qquad (3)$$

Those values of λ for which the boundary value problem (3) has nontrivial solutions $y(x)$ are called the eigenvalues of the boundary value problems (3); the nontrivial solutions are called the associated eigenfunctions.

17.2.1. Theorem: If the boundary-value problem

$$\left.\begin{array}{l} L[y] = 0, \\ V_k(y) = 0 \;\; (k = 1,2, \dots, n) \end{array}\right\} \qquad (4)$$

has the Green's function $G(x, \xi)$, then the boundary value problem $(1) - (2)$ is equivalent to the Fredholm integral equation

$$y(x) = \lambda \int_a^b G(x, \xi) y(\xi)\, d\xi + f(x) \qquad (5)$$

where

$$f(x) = \int_a^b G(x,\xi)h(\xi)\, d\xi \qquad (6)$$

In particular, the homogeneous boundary value problem (3) is equivalent to the homogeneous integral equation

$$y(x) = \lambda \int_a^b G(x,\xi)y(\xi)\, d\xi \qquad (7)$$

17.2.2. Note: Since $G(x,\xi)$ is a continuous kernel, the Fredholm theory is applicable to the integral equation. Therefore, the homogeneous integral equation (7) can have at most a countable number of characteristic numbers $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ which do not have a finite limit point. For all values of λ different from the characteristic values, the nonhomogeneous equation (5) has a solution for any continuous right side $f(x)$. This solution is given by the formula

$$y(x) = \lambda \int_a^b R(x,\xi;\lambda)f(\xi)\, d\xi + f(x) \qquad (8)$$

Where $R(x,\xi;\lambda)$ is the resolvent kernel of the kernel $G(x,\xi)$. Here, for any fixed values of x and ξ in $[a,b]$ the function $R(x,\xi;\lambda)$ is a meromorphic function of λ; only characteristic numbers of the homogeneous integral equation (7) may be the poles of this function.

17.2.3 Example: Reduce the boundary value problem

$$y'' + \lambda y = x, \qquad (1)$$

$$y(0) = y\left(\frac{\pi}{2}\right) = 0 \qquad (2)$$

to an integral equation.

Solution: First, find the Green's function $G(x,\xi)$ for the corresponding homogeneous problem:

$$\left.\begin{aligned} y''(x) &= 0, \\ y(0) = y\left(\frac{\pi}{2}\right) &= 0 \end{aligned}\right\} \qquad (3)$$

Since the function $y_1(x) = x$ and $y_2(x) = x - \frac{\pi}{2}$ are, respectively, linearly independent solutions of the equation $y''(x) = 0$ that satisfy the conditions $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 0$, we seek Green's function in the form

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)\, y_2(\xi)}{W(\xi)}, & 0 \le x \le \xi \\ \dfrac{y_{1}(\xi)\, y_{2}(x)}{W(\xi)}, & \xi \le x \le \frac{\pi}{2} \end{cases}$$

where

$$W(\xi) = \begin{vmatrix} \xi & \xi - \frac{\pi}{2} \\ 1 & 1 \end{vmatrix} = \frac{\pi}{2}$$

Thus,

$$G(x,\xi) = \begin{cases} \left(\frac{2}{\pi}\,\xi - 1\right)x, & 0 \le x \le \xi \\ \left(\frac{2}{\pi}\,x - 1\right)\xi, & \xi \le x \le \frac{\pi}{2} \end{cases} \qquad (4)$$

Further, taking advantage of Green's function (4) as the kernel of an integral equation, we got the following integral equation for $y(x)$:

$$y(x) = f(x) - \lambda \int_0^{\frac{\pi}{2}} G(x,\xi)y(\xi)\,d\xi$$

$$f(x) = \int_0^{\frac{\pi}{2}} G(x,\xi)\xi\,d\xi$$

$$= \int_0^x \left(\frac{2x}{\pi} - 1\right)\xi^2\,d\xi + \int_x^{\frac{\pi}{2}} \left(\frac{2\xi}{\pi} - 1\right)\xi d\xi = \frac{1}{6}x^3 - \frac{\pi^2}{24}x$$

Thus, the boundary value problem$(1) - (2)$ has been reduced to the integral equation

$$y(x) + \lambda \int_0^{\frac{\pi}{2}} G(x,\xi)y(\xi)\,d\xi = \frac{1}{6}x^3 - \frac{\pi^2}{24}x$$

17.2.4 Example: Reduce the boundary value problem $y'' = \lambda y + x^2,\quad y(0) = y\left(\frac{\pi}{2}\right) = 0$ to the integral equation.

Solution: First, we find the Green's function $G(x,\xi)$ for the corresponding homogeneous problem

$$y''(x) = 0\,, \qquad y(0) = 0 = y\left(\frac{\pi}{2}\right)$$

$$y'(x) = C_1$$

$$y(x) = C_1 x + C_2$$

Now $\quad y(0) = C_1(0) + C_2 \implies C_2 = 0$

$y'(0) = C_1 \implies C_1 = 1$

$y_1(x) = x$

$y\left(\frac{\pi}{2}\right) = C_1\left(\frac{\pi}{2}\right) + C_2$

$0 = C_1\left(\frac{\pi}{2}\right) + C_2$

$C_2 = -C_1\left(\frac{\pi}{2}\right)$

$y'\left(\frac{\pi}{2}\right) = C_1 \implies C_1 = 1$

$$y_2(x) = x - \frac{\pi}{2}$$

Wronskian:

$$w(\xi) = \begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{vmatrix}, \quad y_1(\xi) = \xi \implies y_1'(\xi) = 1,$$

$y_2(\xi) = \xi - \dfrac{\pi}{2} \implies y_2'(\xi) = 1.$

$$w(\xi) = \begin{vmatrix} \xi & \xi - \frac{\pi}{2} \\ 1 & 1 \end{vmatrix} = \frac{\pi}{2}$$

Green's Function:

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{w(\xi)} & 0 \le x \le \xi \\ \dfrac{y_1(\xi)y_2(x)}{w(\xi)} & \xi \le x \le \dfrac{\pi}{2} \end{cases}$$

$$= \begin{cases} \dfrac{x(\xi - \frac{\pi}{2})}{\frac{\pi}{2}} & 0 \le x \le \xi \\ \dfrac{\xi(x - \frac{\pi}{2})}{\frac{\pi}{2}} & \xi \le x \le \frac{\pi}{2} \end{cases}$$

$$= \begin{cases} \dfrac{2x\xi}{\pi} - x & 0 \le x \le \xi \\ \dfrac{2x\xi}{\pi} - \xi & \xi \le x \le \dfrac{\pi}{2} \end{cases}$$

Now, we reduce the integral equation to the required equation

$$y(x) = f(x) - \lambda \int_0^{\pi/2} G(x,\xi) \; y(\xi) d\xi$$

Here $f(x) = x^2$

$$y(x) + \lambda \int_0^{\frac{\pi}{2}} G(x,\xi) \; y(\xi) d\xi = \int_0^{\frac{\pi}{2}} G(x,\xi) \; \xi^2 d\xi$$

$$I = \int_0^x G_1(x,\xi)\xi^2 \; d\xi + \int_x^{\pi/2} G_2(x,\xi) \; \xi^2 d\xi$$

$$I_1 = \int_0^x \left(\frac{2x}{\pi} - 1\right) \xi \cdot \xi^2 d\xi$$

$$= \left(\frac{2x}{\pi} - 1\right) \int_0^x \xi^3 \; d\xi$$

$$= \left(\frac{2x}{\pi} - 1\right) \left[\frac{\xi^4}{4}\right]_0^x$$

$$= \left(\frac{2x}{\pi} - 1\right)\left[\frac{x^4}{4}\right]$$

$$I_1 = \frac{x^5}{2\pi} - \frac{x^4}{4}$$

$$I_2 = \int_x^{\frac{\pi}{2}} G_2\,(x,\xi)\xi^2 d\xi$$

$$= \int_x^{\frac{\pi}{2}} \left(\frac{2\xi}{\pi} - 1\right) x\xi^2\, d\xi$$

$$= \frac{2x}{\pi}\int_x^{\pi/2} \xi^3\, d\xi - x\int_x^{\pi/2} \xi^2\, d\xi$$

$$= \frac{2x}{\pi}\left[\frac{\xi^4}{4}\right]_x^{\pi/2} - x\left[\frac{\xi^3}{3}\right]_x^{\pi/2}$$

$$= \frac{2x}{\pi}\left[\frac{\frac{\pi^4}{16}}{4} - \frac{x^4}{4}\right] - x\left[\frac{\frac{\pi^3}{8}}{3} - \frac{x^3}{3}\right]$$

$$I_2 = \frac{x^4}{3} - \frac{x^5}{2\pi} - \frac{\pi^3}{96}x$$

$$I = I_1 + I_2$$

$$I = \frac{x^4}{12} - \frac{\pi^3}{96}x$$

$$y(x) + \lambda\int_0^{\pi/2} G(x,\xi)\;\; y\,(\xi)d\xi = \frac{x^4}{12} - \frac{\pi^3}{96}x.$$

17.2.5 Example: Reduce the boundary value problem $y'' + \lambda y = e^x$, $y(0) = y'(0)$, $y(1) = y'(1)$ to the integral equation.

Solution: First, we find the Green's function $G(x,\xi)$ for the corresponding homogeneous problem

$$y''(x) = 0\;\; , y(0) = y'(0),\; y(1) = y'(1)$$

$$y'(x) = A$$

$$y(x) = Ax + B$$

$$y(0) = A(0) + B \; then \; y(0) = B, \qquad y'(0) = A \; then \; A = B \;\; , \; y_1(x) = x + 1$$

$$y(1) = A + B \; then \; y'(1) = A \; , \; A = A + B \; then \; B = 0, \; y_2(x) = x$$

Wronskian:

$$w(\xi) \;=\; \begin{vmatrix} y_1(\xi) & y_2(\xi) \\ y_1'(\xi) & y_2'(\xi) \end{vmatrix} = \begin{vmatrix} \xi + 1 & \xi \\ 1 & 1 \end{vmatrix} = 1$$

Green's Function:

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(x)}{w(\xi)} & 0 \le x \le \xi \\ \dfrac{y_1(\xi)y_2(\xi)}{w(\xi)} & \xi \le x \le 1 \end{cases}$$

$$G(x,\xi) \quad = \quad \begin{cases} (x+1)\ \xi & 0 \le x \le \xi \\ (\xi+1)\,x & \xi \le x \le 1 \end{cases}$$

Now, we reduce to an integral equation

$y(x) = f(x) - \lambda \int_0^1 G(x,\xi)y(\xi)\,d\xi,$

$f(x) = e^x$

$$I = \int_0^1 G(x,\xi)e^{\xi}\,d\xi$$

$I = \int_0^x G_1(x,\xi)e^{\xi}d\xi + \int_x^1 G_2\,(x,\xi)e^{\xi}d\xi$

$$I_1 = \int_0^x x(\xi+1)\xi\, e^{\xi}d\xi$$

$= x\left[\int_0^x \xi e^{\xi}\,d\xi + \int_0^x e^{\xi}\,d\xi\right]$

$= x\left[(\xi e^{\xi} - e^{\xi}\right]_0^x + \left[e^{\xi}\right]_0^x$

$= x[xe^x]$

$I_1 = x^2\,e^x$

$$I_2 = \int_x^1 \xi(x+1)e^{\xi}d\xi = (x+1)\int_x^1 \xi e^{\xi}\,d\xi$$

$= (x+1)\left[(\xi e^{\xi} - e^{\xi}\right]_x^1$

$= (x+1)[-xe^x + e^x]$

$= -x^2e^x + e^x$

$I = I_1 + I_2 = e^x$

$y(x) + \lambda \int_0^1 G(x,\xi)y(\xi)\,d\xi = e^x.$

17.3 SUMMARY:

In this section, we learnt about converting the BVPs containing a numerical parameter λ to a Fredholm integral equation with the help of Green's function. A few examples and the fundamental theorem related to BVPs with Green's function, which is equivalent to the Fredholm integral equation, were discussed for the better understanding of the reader.

17.4 TECHNICAL TERMS:

Green's function, Fredholm integral equation.

17.5 SELF-ASSESSMENT QUESTIONS:

Reduce the following boundary value problems to integral equations.

1. $y'' + \lambda y = 2x + 1\,,\quad y(0) = y'(1),\ y'(0) = y(1).$
2. $y'''' = \lambda y + 1\,,\quad y(0) = y'(0) = 0,\ y''(1) = y'''(1) = 0.$
3. $y'' + \frac{\pi^2}{4} y = \lambda y + cos\frac{\pi x}{2},\quad y(-1) = y(1),\ y'(-1) = y'(1).$
4. $y''' + \lambda y = 2x\,,\quad y(0) = y(1) = 0,\ y'(0) = y'(1).$

Answers to Self-Assessment Questions:

1. $y(x) = -\lambda \int_0^1 G(x,\xi)\, y(\xi)\, d\xi + \frac{1}{6}(2x^3 + 3x^2 - 17x - 5)\,,$

$$G(x,\xi) = \begin{cases} (\xi - 2)x + \xi - 1, & 0 \le x \le \xi \\ (\xi - 1)x - 1, & \xi \le x \le 1 \end{cases}$$

2. $y(x) = \lambda \int_0^1 G(x,\xi)\, y(\xi)\, d\xi + \frac{x^2}{24}(x^2 - 4x + 6),$

$$G(x,\xi) = \begin{cases} \frac{x^2}{6}(3\xi - x), & 0 \le x \le \xi \\ \frac{\xi^2}{6}(3x - \xi), & \xi \le x \le 1 \end{cases}$$

3. $y(x) = \lambda \int_{-1}^1 G(x,\xi)\, y(\xi)\, d\xi + \frac{x}{\pi}\ sin\frac{\pi x}{2} + \frac{2}{\pi^2}\cos\frac{\pi x}{2}\,,$

$$G(x,\xi) = \begin{cases} \frac{1}{\pi}\sin\frac{\pi}{2}(\xi - x\,), & -1 \le x \le \xi \\ \frac{1}{\pi}\sin\frac{\pi}{2}\ (x - \xi), & \xi \le x \le 1 \end{cases}$$

4. $y(x) = -\lambda \int_0^1 G(x,\xi)\, y(\xi) d\xi + \frac{1}{12} x(x-1)(x^2 + x - 1)\,,$

$$G(x,\xi) = \begin{cases} \frac{1}{2}x(x - \xi)(\xi - 1), & 0 \le x \le \xi \\ -\frac{1}{2}\xi(\xi - x)(x - 1), & \xi \le x \le 1 \end{cases}$$

17.6 SUGGESTED READINGS:

1. M. D. Raisinghania, Integral equations and Boundary Value Problems, S. Chand and Company Pvt. Ltd., 2007.
2. Shanti Swarup, Integral equations, Krishna Prakashan Pvt Ltd, Meerut, 2003.
3. M. Krasnov, A. Kiselev, G Makarenko, Problems and Exercises in Integral Equations, MIR Publishers, Moscow, 1971.
4. M. Rahman, Integral equations and their applications, WIT Press, Southampton, Boston, 2007.

- **Dr. Madhusmita Tripathy**

LESSON- 18

SINGULAR INTEGRAL EQUATIONS

OBJECTIVE:

- To learn about singular integral equations.
- Finding eigenvalues and eigen functions of singular integral equations.

STRUCTURE:

18.1 Introduction

18.2 Eigenvalues and Eigenfunctions of Singular Integral Equation

18.3 Summary

18.4 Technical Terms

18.5 Self-Assessment Questions

18.6 Suggested Readings

18.1 INTRODUCTION:

This section is concerned with singular integral equations that has enormous applications in problems including fluid mechanics, bio-mechanics, and electromagnetic theory. An integral equation is called a singular integral equation if one or both limits of integration becomes infinite, or if the kernel K (x, t), of the equation becomes infinite at one or more points in the interval of integration.

18.2 EIGENVALUES AND EIGENFUNCTIONS OF SINGULAR INTEGRAL EQUATION:

The following integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\, \varphi(t)\, dt \qquad (1)$$

is singular if the interval of integration (a, b) is infinite or the kernel $K(x,t)$ is non-integrable in the sense of L_2 (Ω).

In case of singular integral equations, if the kernel $K(x,t)$ is continuous in $\Omega\,\{\, a \le x, t \le b\}$ and a and b are finite, then the spectrum of the integral equation is the set of characteristic numbers and for every characteristic number there corresponds at most a finite number of linearly independent eigenfunctions (the characteristic numbers can have a finite multiplicity). For singular Integral equations, the spectrum may be continuous. It means that, the characteristic numbers may fill the whole intervals, and there may be characteristic numbers of infinite multiplicity. We will discuss this through an example.

Considering the Lalesco-Picard equation

$$\varphi(x) = \lambda \int_{-\infty}^{+\infty} e^{-|x-t|}\, \varphi(t)\, dt \qquad (2)$$

The kernel of this equation, $K(x,t) = e^{-|x-t|}$, possesses an infinite norm,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K^2(x,t)\, dx\, dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-2|x-t|}\, dx\, dt = \int_{-\infty}^{+\infty} dx$$

If the function $\varphi(x)$ is twice differentiable, then the integral equation (2), can be written in the form

$$\varphi(x) = \lambda \left[e^{-x} \int_{-\infty}^{x} e^{t}\, \varphi(t)\, dt + e^{x} \int_{x}^{+\infty} e^{-t}\, \varphi(t)\, dt \right]$$

which is equivalent to the differential equation

$$\varphi''(x) + (2\lambda - 1)\varphi(x) = 0 \qquad (3)$$

The general solution of equation (3) is of the form

$$\varphi(x) = C_1 e^{rx} + C_2 e^{-rx} \qquad (4)$$

with C_1, C_2 arbitrary constants, and

$$r = \sqrt{1 - 2\lambda} \qquad (5)$$

For the integral in the right hand side of equation (2) to exist, it is necessary that $|Re\, r| < 1$, that is, $\lambda > 0$ for real λ. Hence, in the domain of real numbers the spectrum of equation (2) fills the infinite interval $0 < \lambda < +\infty$. Every point of this interval is a characteristic number of equation (2) of multiplicity 2. However, the associated eigenfunctions do not belong to the class $L_2(-\infty, +\infty)$.

It follows from equation (4) that for $\lambda > \frac{1}{2}$, the eigenfunctions are

$\sin\sqrt{2\lambda - 1}\, x$, $\cos\sqrt{2\lambda - 1}\, x$.

For $\lambda = \frac{1}{2}$, we obtain $\varphi(x) = C_1 + C_2\, x$. Thus, we conclude that for $\lambda \geq \frac{1}{2}$, the eigenfunctions are bounded in $(-\infty, +\infty)$.

However, if the real part $\sqrt{1 - 2\lambda}$ is positive and less than unity, then formula (4) is valid, for any choice of the constants, $C_1, C_2\ (C_1^2 + C_2^2 \neq 0)$ solution of the integral equation (2) unbounded on $(-\infty, +\infty)$.

The above example illustrates the essential role of the class of functions in which the solution of the integral equation is sought.

Thus, if we seek the solution of equation (2) in the class of bounded functions, then values of $\lambda > \frac{1}{2}$ are characteristic.

But if the solution of equation (2) is sought in the class of $L_2(-\infty, +\infty)$ functions. Then, for any values of λ equation (2) has only the trivial solution $\varphi(x) \equiv 0$, i.e., not one of the values of λ is characteristic for solution in $L_2(-\infty, +\infty)$.

Let $F(x)$ be a continuous function which is absolutely integrable on $[0, +\infty]$ and having a finite number of maxima and minima on any finite interval of the x-axis.

Constructing the Fourier cosine transform of this functions:

$$F_1(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F(x) \cos \lambda x \; dx$$

Then

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} F_1(\lambda) \; \cos \lambda x \; d\lambda$$

Adding these two formulas, we get

$$F_1(x) + F(x) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} [F_1(t) + F(t)] \; \cos xt \;\; dt$$

That is, for any choice of the function $F(x)$ satisfying the above indicated conditions, the function $\varphi(x) = F_1(x) + F(x)$ is an eigenfunction of the integral equation

$$\varphi(x) = \lambda \int_0^{+\infty} \varphi(t) \cos xt \; dt \qquad (6)$$

corresponding to the characteristic value $\lambda = \sqrt{\frac{2}{\pi}}$.

Since $F(x)$ is an arbitrary function, it follows that for the indicated value of λ, as given in sequation (6) has an infinite number of linearly independent eigenfunctions.

18.2.1 Example: Consider the integral equation

$$\varphi(x) = \lambda \int_0^{\infty} \varphi(t) \; \cos xt \; dt \qquad (7)$$

Taking $F(x) = e^{-ax}$ $(a > 0)$, Then $F_1(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at} \cos xt \; dt = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}$.

So, $\varphi(x) = F(x) + F_1(x) = e^{-ax} + \sqrt{\frac{2}{\pi}} \frac{a}{a^2+x^2}$ (8)

Substituting $\varphi(x)$ int equation (7), we have

$$e^{-ax} + \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2} = \lambda \left[\int_0^{\infty} e^{-at} cos\; xt\; dt + \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{a\; cos\;\; xt}{a^2+t^2} \; dt \right] \qquad (9)$$

As already has been pointed out

$$\int_0^{\infty} e^{-at}\, cos\; xt \;\; dt = \frac{a}{a^2+x^2}$$

The second integral on the right of (9) may be found by using Cauchy's theorem on residues:

$$\int_0^{\infty} \frac{cos\ xt}{a^2 + t^2}\ dt\ = \frac{\pi}{2a} e^{-ax}$$

From (9) we thus obtain

$$e^{-ax} + \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2} = \lambda \left[\frac{a}{a^2 + x^2} + \sqrt{\frac{\pi}{2}} e^{-ax} \right] \tag{10}$$

If $\lambda = \sqrt{\frac{2}{\pi}}$, then the function

$$\varphi(x) = e^{-a} + \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2} \not\equiv 0$$

will be a solution of the integral equation (7). Hence, $\lambda = \sqrt{\frac{2}{\pi}}$ is a characteristic number of (7), and the function

$$\varphi(x) = e^{-a} + \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2} \tag{8}$$

is the corresponding eigenfunction. Since a is any number greater than 0, the characteristic number $\lambda = \sqrt{\frac{2}{\pi}}$ is associated with an infinity of linearly independent eigenfunctions (8). Similarly, we can show that equation (7) has a characteristic number $\lambda = -\sqrt{\frac{2}{\pi}}$ associated with the eigenfunctions

$$e^{-ax} - \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2} \qquad (a > 0)$$

18.2.2 Example: Show that the integral equation $\varphi(x) = \lambda \int_0^{\infty} \varphi(t)\ sinxt\ dt$ has characteristic number $\lambda = \pm\sqrt{\frac{2}{\pi}}$ of infinite multiplicity and find the associated eigen functions.

Solution: Let us consider the integral equation

$$\varphi(x) = \lambda \int_0^{\infty} \varphi(t)\ sinxt\ dt \tag{1}$$

We take $F(x) = e^{-ax} \quad (a > 0)$, then $F_1(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-at}\ sinxt\ dt = \sqrt{\frac{2}{\pi}} \frac{x}{a^2+x^2}$

Further, $\varphi(x) = F(x) + F_1(x) = e^{-ax} + \sqrt{\frac{2}{\pi}} \frac{x}{a^2+x^2}$.

Substituting $\varphi(x)$ into equation (1), we have

$$e^{-ax} + \sqrt{\frac{2}{\pi}} \cdot \frac{x}{a^2+x^2} = \lambda\left[\int_0^{\infty} e^{-at}\ sinxt\ dt\ + \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{t\ sinxt}{a^2+x^2}\ dt\right]$$

We know that the first term $\int_0^{\infty} e^{-at}\ \sin xt\ dt\ = \frac{x}{a^2+x^2}$.

Second term (by standard integral table $\int_0^{\infty}\frac{t\ sinxt}{a^2+t^2} = \frac{\pi}{2}e^{-ax}$.

Thus, $e^{-ax} + \sqrt{\frac{2}{\pi}}\frac{x}{a^2+x^2} = \lambda\left[\frac{x}{a^2+x^2} + \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}e^{-ax}\right]$.

i.e., $e^{-ax} + \sqrt{\frac{2}{\pi}}\frac{x}{a^2+x^2} = \lambda\left[\frac{x}{a^2+x^2} + \sqrt{2\pi} \cdot \frac{1}{2}e^{-a}\ \right]$.

Now comparing coefficient of e^{-ax} on both the sides, we have $\lambda \cdot \frac{\sqrt{2\pi}}{2} = 1$. i.e., $\lambda = \sqrt{\frac{2}{\pi}}$.

Similarly, comparing the coefficient of $\frac{x}{a^2+x^2}$ on both the sides, we obtain $\lambda = \sqrt{\frac{2}{\pi}}$. So, we conclude that the chosen function $\varphi(x)$ satisfies the integral equation exactly when $\lambda = \sqrt{\frac{2}{\pi}}$.

Hence $\lambda = \sqrt{\frac{2}{\pi}}$ is a characteristic number and the function $\varphi(x) = e^{-ax} + \sqrt{\frac{2}{\pi}}\frac{x}{a^2+x^2} \not\equiv 0$ will be the solution of the integral equation (1).

In a similar manner, we can show that equation (1) has a characteristic number $\lambda = -\sqrt{\frac{2}{\pi}}$ associated with the eigenfunctions $e^{-ax} - \sqrt{\frac{2}{\pi}}\frac{x}{a^2+x^2}$.

18.2.3 Example: Show that the integral equation $\varphi(x) = \lambda\int_0^{\infty} J_v\left(2\sqrt{xt}\right)\varphi(t)dt$ has characteristic number $\lambda = \pm 1$ of infinite multiplicity and find the associated eigenfunctions. [where $J_v(z)$ is a Bessel function of the first kind.]

Solution: Given $\varphi(x) = \lambda\int_0^{\infty} J_v\,(2\sqrt{xt})\ \varphi(t)dt$

Using the orthogonality result for the Hankel Kernel

$$\int_0^{\infty} J_v\left(2\sqrt{xt}\right)\sqrt{t}\,J_v\left(2\sqrt{ty}\right)dt = \delta(x-y)$$

This means that $\varphi(t) = \sqrt{t}\,J_v\left(2\sqrt{at}\right)$ are eigenfunctions of the operator

$T(\varphi)(x) = \int_0^{\infty} J_v\left(2\sqrt{xt}\right)\varphi(t)dt$ with the corresponding eigenvalue $\lambda = 1$.

As $T(\sqrt{t}\,J_v\left(2\sqrt{at}\right) = \sqrt{x}\,J_v(2\sqrt{ax})$ and we define $\varphi(t) = \sqrt{t}\,, J_v\left(2\sqrt{at}\right),\ a > 0$

So, $\int_0^{\infty} J_v(2\sqrt{xt})\ \varphi(t)dt = \sqrt{x}\,J_v\left(2\sqrt{ax}\ \right) = \varphi(x)$.

Hence $\varphi(x) = \int_0^{\infty} J_v\left(2\sqrt{xt}\right)\varphi(t)dt$ for $\lambda = 1$. As the integral operator is self-adjoint, the negative eigenvalue is symmetric for real K . If $\lambda = 1$ is an eigen value, then $\lambda = -1$ is also an eigenvalue. An the corresponding eigenfunction is $\varphi(t) = \sqrt{t}\, J_v\left(2\sqrt{at}\right), \ \forall\, a > 0.$

18.3 SUMMARY:

In this section, we learnt about singular integral equations. Here, we try to find eigenvalues and eigenfunctions of singular integral equations. Few examples are discussed for the better understanding of the reader. In computing the eigenfunctions we take the help pf Fourier transform and Cauchy integral equations.

18.4 TECHNICAL TERMS:

Eigenvalues, Eigen functions, Hankel Kernel, Spectrum, Fourier transform, Cauchy integral equation.

18.5 SELF-ASSESSMENT QUESTIONS:

1. Show that for the integral equation $\varphi(x) = \lambda \int_x^{\infty} \frac{(x-t)^n}{n!}\varphi(t)dt,$ any number λ for which one of the values $\sqrt[n+1]{\lambda}$ has a positive real part is a characteristic number.
2. Show that the Volterra integral equation $\varphi(x) = \lambda \int_0^{x} \left(\frac{1}{t} - \frac{1}{x}\right)\varphi(t)dt$, has an infinity of characteristic numbers $\lambda = \xi + i\eta$, where the point (ξ, η) lies outside the parabola $\xi + \eta^2 = 0.$

18.6 SUGGESTED READINGS:

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LESSON- 19

SOLUTION OF SINGULAR INTEGRAL EQUATIONS

OBJECTIVE:

- To learn about the solution of singular integral equations.
- Use Efros theorem as generalized product rule to get solution of singular integral equation.
- To use Mellin Transform for solution of certain singular integral equations.

STRUCTURE:

19.1 Introduction

19.2 Efros Rule for Singular Integral Equations

19.3 Mellin Transform Method for Singular Integral Equation

19.4 Summary

19.5 Technical Terms

19.6 Self-Assessment Questions

19.7 Suggested Readings

19.1 INTRODUCTION:

In this section, we will learn about two important techniques for solving a certain type of singular integral equations. The first type uses the generalized product rule by Efros to obtain the solution of the singular integral equation. The second method is based on Mellin transform to extract the solution of the singular integral equation.

19.2 EFROS RULE FOR SINGULAR INTEGRAL EQUATIONS:

19.2.1 Theorem (**Generalized product rule by Efros**):

Let $\varphi(x) \risingdotseq \Phi(p),\ u(x,\tau) \risingdotseq U(p)e^{-\tau q(p)}$,

where $U(p)$ and $q(p)$ are analytic functions.

Then $\Phi\big(q,(p)\big)U(p) \risingdotseq \int_0^\infty \varphi(\tau)u(x,\tau)d\tau$ (1)

If $u(x,\tau) = u(x-\tau)$, then $q(p) \equiv p$ and we obtain the ordinary product

$$\Phi(p)U(p) \risingdotseq \int_0^\infty \phi(\tau)u(x-\tau)d\tau$$

If $U(p) = \frac{1}{\sqrt{p}},\ q(p) = \sqrt{p}$, then $u(x,\tau) = \frac{1}{\sqrt{\pi x}}e^{\frac{-\tau^2}{4x}}$. (2)

Therefore, if $\Phi(p) \fallingdotseq \varphi(x)$, then by the Efro's theorem, we find the original function for $\frac{\Phi(\sqrt{p})}{\sqrt{p}}$ as

$$\frac{\Phi(\sqrt{p})}{\sqrt{p}} \fallingdotseq \frac{1}{\sqrt{\pi x}}\int_0^{\infty} \varphi(\tau)e^{\frac{-\tau^2}{4x}}\,d\tau \qquad (3)$$

19.2.2 Example: Solve the integral equation

$$\frac{1}{\sqrt{\pi x}}\int_0^{\infty} e^{\frac{-t^2}{4x}}\varphi(t)dt = 1 \qquad (4)$$

Solution: Let $\varphi(x) \fallingdotseq \Phi(\mathrm{p})$. Taking the Laplace transform of both sides of equation (4), we get, by formula (3),

$$\frac{\Phi(\sqrt{p})}{\sqrt{p}} = \frac{1}{p}$$

Whence

$$\frac{\Phi(\mathrm{p})}{p} = \frac{1}{p^2}, \quad \text{or} \quad \Phi(\mathrm{p}) = \frac{1}{p} \fallingdotseq 1$$

Hence, $\varphi(x) \equiv 1$ is a solution of equation (4).

19.2.3 Example: Solve the integral equation $\frac{1}{\sqrt{\pi x}}\int_0^{\infty} e^{\frac{-t^2}{4x}}\varphi(t)dt = e^{-x}$

Solution: Let $\varphi(x) = \Phi(p)$

Given $\frac{1}{\sqrt{\pi x}} e^{\frac{-t^2}{4x}}\varphi(t)dt = e^{-x}$ (5)

Taking the Laplace transform on both sides of equation (5), we obtain

$$L\left[\frac{1}{\sqrt{\pi x}} e^{\frac{-t^2}{4x}}\varphi(t)dt\right] = L[e^{-x}]$$

But we know that $\frac{\Phi(\sqrt{p})}{\sqrt{p}} = \frac{1}{\sqrt{\pi x}}\int_0^{\infty} e^{\frac{-t^2}{4x}}\varphi(t)dt$ and $L[e^{-t}] = \frac{1}{s+1}$

$$\frac{\Phi(\sqrt{p})}{\sqrt{p}} = \frac{1}{s+1}$$

S= $\sqrt{p} \Rightarrow p = s^2$

$\frac{\Phi(p)}{p} = \frac{1}{p^2+1^2} = cosx$, [$\because L^{-1}\left[\frac{S}{S^2+a^2}\right] = cosat$]

So, $\varphi(x) = cosx$ is the required solution.

19.2.4 Example: Solve the integral equation $\frac{1}{\sqrt{\pi x}}\int_0^{\infty} e^{\frac{-t^2}{4x}}\varphi(t)dt = 2x - sinhx$

Solution: Let $\phi(x) = \Phi(p)$

Given $\frac{1}{\sqrt{\pi x}}\int_0^\infty e^{\frac{-t^2}{4x}}\varphi(t)dt = 2x - sinhx$ (6)

Taking Laplace transform on both the sides of equation (6), we obtain

$$L\left[\frac{1}{\sqrt{\pi x}}\int_0^\infty e^{\frac{-t^2}{4x}}\varphi(t)dt\right] = L[2x - sinhx\,]$$

$$\frac{\Phi(\sqrt{p})}{\sqrt{p}} = \frac{2}{p^2} - \frac{1}{p^2-1}$$

$$\frac{\Phi(p)}{p} = \frac{2}{p^4} - \frac{1}{p^4-1}$$

$$\Phi(p) = \frac{2p}{p^4} - \frac{p}{p^4-1}$$

$$\Phi(p) = \frac{2}{p^3} - \frac{p}{p^4-1}$$

$$\Phi(p) = \frac{2}{p^3} - \frac{p}{(p^2-1)(p^2+1)}$$

Consider $\frac{p}{(p^2-1)(p^2+1)} = \frac{Ap+B}{(p^2-1)} + \frac{Cp+D}{(p^2+1)}$

$p = (Ap+B)(p^2+1) + (Cp+D)\,(p^2-1)$

$p = (A+C)p^3 + (A-C)p + (B+D)p^2 + B - D$

$$= \left[\frac{2}{p^3} - \frac{p/2}{(p^2-1)} - \frac{p/2}{(p^2+1)}\right]$$

$= x^2 - \frac{1}{2}(coshx - cosx)$, which is the desired solution.

19.2.5 Note: It is known that

$$t^{\frac{n}{2}}J_n(2\sqrt{t}) \risingdotseq \frac{1}{p^{n+1}}e^{-\frac{1}{p}} \quad (n = 0,1,2,\ldots.) \qquad (7)$$

Where $J_n(z)$ is a Bessel function of the first kind of order n. In particular,

$$J_0(2\sqrt{t}) \risingdotseq \frac{1}{p}e^{-\frac{1}{p}} \qquad (8)$$

By virtue of the similarity theorem

$$J_0(2\sqrt{xt}) \risingdotseq \frac{1}{p}e^{-\frac{x}{p}} \qquad (9)$$

It follows from Efro's theorem that $q(p) \equiv \frac{1}{p}$.

19.2.6 Example: Solve the integral equation

$$\varphi(x) = xe^{-x} + \lambda \int_0^\infty J_0(2\sqrt{xt})\varphi(t)dt \qquad (|\lambda| \neq 1) \qquad (10)$$

Solution: Let $\varphi(x) \risingdotseq \Phi(\mathrm{p})$. Taking the Laplace transform of both sides of (10) and considering the Efros theorem, we find

$$\Phi(\mathrm{p}) = \frac{1}{(p+1)^2} + \lambda \frac{1}{p} \Phi\left(\frac{1}{p}\right) \qquad (11)$$

Replacing p by $\frac{1}{p}$, we get

$$\Phi\left(\frac{1}{p}\right) = \frac{p^2}{(p+1)^2} + \lambda p \Phi(\mathrm{p}) \qquad (12)$$

From equation (11) and (12), we find

$$\Phi(\mathrm{p}) = \frac{1}{(p+1)^2} + \frac{\lambda}{p}\left[\frac{p^2}{(p+1)^2} + \lambda p \Phi(\mathrm{p})\right]$$

or

$$\Phi(\mathrm{p}) = \frac{1}{1-\lambda^2}\left[\frac{1}{(p+1)^2} + \frac{\lambda p}{(p+1)^2}\right]$$

Hence, $\varphi(x) = e^{-x}\left(\frac{x}{1+\lambda} + \frac{\lambda}{1-\lambda^2}\right)$ is the desire solution.

19.2.7 Example: Solve the following integral equation

$$\varphi(x) = cosx + \lambda \int_0^\infty J_0\left(2\sqrt{xt}\right)\varphi(t)dt$$

Solution: Given that $\varphi(x) = cosx + \lambda \int_0^\infty J_0\left(2\sqrt{xt}\right)\varphi(t)dt$ (13)

Taking Laplace transform of both sides in equation (13), we obtain

$$\Phi(p) = \frac{p}{p^2+1} + \lambda \frac{1}{p} \Phi\left(\frac{1}{p}\right) \qquad (14)$$

Replacing p by $1/p$ in (14), we obtain

$$\Phi\left(\frac{1}{p}\right) = \frac{1/p}{1/p^2+1} + \lambda p\, \Phi(p),$$

$$\Phi\left(\frac{1}{p}\right) = \frac{p}{p^2+1} + \lambda p\, \Phi(p) \qquad (15)$$

Now using equation (15) in (14), we have

$$\Phi(p) = \frac{p}{p^2+1} + \lambda \frac{1}{p}\left[\frac{p}{1+p^2} + \lambda p\, \Phi(p)\right]$$

$$\Phi(p) = \frac{p}{p^2+1} + \frac{\lambda}{1+p^2} + \lambda^2\Phi(p)$$

$$(1-\lambda^2)\Phi(p) = \frac{p}{p^2+1} + \frac{\lambda}{1+p^2}$$

Applying inverse Laplace Transform, we obtain the require solution as,

$$\varphi(x) = \frac{1}{1-\lambda^2}[\cos x \ + \lambda\, sinx\,].$$

19.2.8 Example: Solve the following integral equation

$$\varphi(x) = sinx + \lambda\int_0^\infty \sqrt{\frac{x}{t}}J_1(2\sqrt{xt}\,)\varphi(t)\,dt$$

Solution: Given that

$$\varphi(x) = sinx + \lambda\int_0^\infty \sqrt{\frac{x}{t}}J_1(2\sqrt{xt}\,)\varphi(t)\,dt \qquad (16)$$

Applying Laplace transform to both the sides, we obtain

$$\Phi(p) = \frac{1}{p^2+1} + \lambda\left(\frac{1}{p^2}\right)\Phi\left(\frac{1}{p}\right) \qquad (17)$$

Replace p by $1/p$ in equation (2), we get

$$\Phi\left(\frac{1}{p}\right) = \frac{1}{\frac{1}{p^2+1}} + \lambda p^2\Phi(p) = \frac{p^2}{p^2+1} + \lambda p^2\Phi(p) \qquad (18)$$

Using equation (18) in (17), we obtain

$$\Phi(p) = \frac{1}{p^2+1} + \lambda\left(\frac{1}{p^2}\right)\left[\frac{p^2}{p^2+1} + \lambda p^2\Phi(p)\right]$$

$$\Phi(p) = \frac{1}{p^2+1} + \frac{\lambda}{p^2+1} + \lambda^2\Phi(p)$$

$$\Phi(p) - \lambda^2\,\Phi(p) = (\lambda+1)\left(\frac{1}{p^2+1}\right),$$

$$(1-\lambda^2)\Phi(p) = (\lambda+1)\left(\frac{1}{p^2+1}\right).$$

Applying inverse Laplace transform on both sides of the above equation, we get

$\varphi(x) = \frac{(\lambda+1)}{(1-\lambda^2)}sinx = \frac{sinx}{1-\lambda}$ as the require solution of the integral equation.

19.3 MELLIN TRANSFORM METHOD FOR SINGULAR INTEGRAL EQUATION:

Let a function $f(t)$ be defined for positive t and let it satisfy the conditions

$$\int_0^1 |f(t)|\, t^{\sigma_1 - 1}\, dt < +\infty, \qquad \int_1^\infty |f(t)|\, t^{\sigma_2 - 1}\, dt + \infty \tag{1}$$

for a proper choice of the numbers σ_1 and σ_2. The function

$$F(s) = \int_0^\infty f(t) t^{s-1} dt \; (s = \sigma + i\tau, \sigma_1 < \sigma < \sigma_2) \tag{2}$$

is the Mellin transform of the function $f(t)$. The inversion formula of the Mellin transformation is

$$f(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s) t^{-s} ds \;\; (t > 0, \sigma_1 < \sigma < \sigma_2) \tag{3}$$

Where the integral is taken along the straight line l: $Res = \sigma$ parallel to the imaginary axis of the s plane and is understood to be the principal value. When the behaviour of the function $f(t)$ as $t \longrightarrow 0$ and $t \longrightarrow \infty$ is known, say from physical reasoning, then the boundaries of the strip (σ_1, σ_2) may be established from the conditions of the absolute convergence of the integral (2). But if the behaviour of $f(t)$ is only known at one end of the interval $(0, +\infty)$, say as $t \longrightarrow 0$, then only σ_1 is defined, the straight line of integration l in (3) must be chosen to the right of the straight line $\sigma = \sigma_1$ and to the left of the closest singularity of the function $F(s)$.

The Mellin transformation is closely associated with the transformations of Fourier and Laplace, and many theorems which refer to the Mellin transformation can be obtained from the corresponding theorems for the Fourier and Laplace transformations by means of a change of variables. The convolution theorem for the Mellin transformation is of the form

$$M\left\{\int_0^\infty f(t)\varphi\left(\frac{x}{t}\right)\frac{dt}{t}\right\} = F(s).\Phi(s) \tag{4}$$

From this, we can conclude that the Mellin transformation is convenient for the solution of integral equations of the form

$$\varphi(x) = f(x) + \int_0^\infty K\left(\frac{x}{t}\right)\varphi(t)\frac{dt}{t} \tag{5}$$

Let the function $\varphi(x)$, $f(x)$ and $K(x)$ admit the Mellin transformation, and let $\varphi(x) \longrightarrow \Phi(s)$, $f(x) \longrightarrow F(s), K(x) \longrightarrow \widetilde{K}(s)$; the domains of analyticity of $F(s)$ and $\widetilde{K}(s)$ have a common strip $\sigma_1 < \sigma < \sigma_2$. Taking the Mellin transform of both sides of equation (5) and utilizing the convolution theorem (4), we obtain

$$\Phi(s) = F(s) + \widetilde{K}(s) \cdot \Phi(s) \tag{6}$$

Whence

$$\Phi(s) = \frac{F(s)}{1 - \widetilde{K}(s)}\left(\widetilde{K}(s) \neq 1\right) \tag{7}$$

This is the operator solution of the integral equation (5). Using the inverse formula (3), we find the solution $\varphi(x)$ of this equation:

$$\varphi(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{F(s)}{1-\widetilde{K}(s)} x^{-s} ds \qquad (8)$$

19.3.1 Example: Solve the integral equation

$$\varphi(x) = e^{-\alpha x} + \frac{1}{2}\int_0^{\infty} e^{-\frac{x}{t}}\, \varphi(t)\frac{dt}{t} \quad (\alpha > 0) \qquad (9)$$

Solution: Applying the Mellin transform to both sides of equation (9), we obtain

$$M\{e^{-\alpha x}\} = \int_0^{\infty} e^{-\alpha x}\, x^{s-1} dx = \alpha^{-s}\int_0^{\infty} e^{-z} z^{s-1}\, dz = \frac{\Gamma(s)}{\alpha^s} \equiv F(s),$$

$$M\left\{\frac{1}{2}e^{-x}\right\} = \frac{1}{2}\Gamma(s) \equiv \widetilde{K}(s)(Res > 0)$$

so that the domain of analyticity of $F(s)$ and $\widetilde{K}(s)$ coincide. The operator equation corresponding to equation (9) will have the form

$$\Phi(s) = \frac{\Gamma(s)}{\alpha^s} + \frac{1}{2}\Gamma(s)\Phi(s) \qquad (10)$$

Whence

$$\Phi(s) = \frac{\Gamma(s)}{\alpha^s\left[1-\frac{1}{2}\Gamma(s)\right]}$$

Using the inverse formula (8) we obtain

$$\varphi(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s)}{1-\frac{1}{2}\Gamma(s)}\cdot\frac{ds}{(\alpha x)^s} (\sigma > 0) \qquad (11)$$

We find the integral (11) with the aid of Cauchy's integral formula.

For $\alpha x > 1$, we include in the contour of integration the semicircle lying in the right half-plane. In this case, the sole singularity of the integrand lies at the point $s = 3$ at which

$$1 - \frac{1}{2}\,\Gamma(s) = 0$$

Then

$$\varphi(x) = \frac{2}{(\alpha x)^3\psi(3)}, \qquad \alpha x > 1$$

Where $\psi(3)$ is the logarithmic derivative of the Γ function at the point $s = 3$:

$$\psi(3) = \frac{\Gamma'(3)}{\Gamma(3)} = \frac{3}{2} - \gamma\,,\ \gamma \text{ is Euler's constant.}$$

For $\alpha x < 1$, the singularities of the integrand are the negative roots of the function $1 - \frac{1}{2}\,\Gamma(s)$, so that

$$\varphi(x) = -2\sum_{k=1}^{\infty} \frac{1}{(\alpha x)^{s_k}\,\psi(s_k)} \quad , \quad \alpha x < 1$$

where $\psi(s_k)$ are values of the logarithmic derivative $\Gamma(s)$ at the points $s = s_k$ $(k = 1,2,\dots)$.,

Finally, we obtain the solution as,

$$\varphi(x) = \begin{cases} \dfrac{4}{(3-2\gamma)(\alpha x)^3}, & \alpha x > 1, \\ -2\displaystyle\sum_{k=1}^{\infty} \frac{1}{(\alpha x)^{s_k}\,\psi(s_k)}, & \alpha x < 1 \end{cases}$$

19.3.2 Example: Solve the integral equation

$$\varphi(x) = f(x) + \int_0^{\infty} K(x,t)\varphi(t)\,dt \qquad (12)$$

Solution: Multiplying both sides of (12) by x^{s-1} and integrating with respect to x between the limits 0 and ∞, we get

$$\int_0^{\infty} \varphi(x)x^{s-1}dx = \int_0^{\infty} f(x)x^{s-1}dx + \int_0^{\infty} \varphi(t)dt \int_0^{\infty} K(xt)\,x^{s-1}dx$$

Denoting the Mellin transform of the functions $\varphi(x)$, $f(x), K(x)$ by $\Phi(\mathrm{s})$, $F(s)$, $\widetilde{K}(s)$, respectively, we obtain,

$$\Phi(\mathrm{s}) = F(s) + \widetilde{K}(s)\int_0^{\infty} \varphi(t)\,t^{-s}dt \qquad (13)$$

It is easy to see that $\int_0^{\infty} \varphi(t)\,t^{-s}dt = \Phi(1-\mathrm{s})$ so that equation (13) will be written in the form

$$\Phi(\mathrm{s}) = F(s) + \Phi(1-\mathrm{s})\widetilde{K}(s) \qquad (14)$$

Replacing s by $1-s$ in (14), we get

$$\Phi(1-\mathrm{s}) = F(1-s) + \Phi(\mathrm{s})\widetilde{K}(1-s) \qquad (15)$$

From (14) and (15) we find

$$\Phi(\mathrm{s}) = F(s) + F(1-s)\widetilde{K}(s) + \Phi(\mathrm{s})\widetilde{K}(s)\cdot\,\widetilde{K}(1-s)$$

Whence

$$\Phi(\mathrm{s}) = \frac{F(s) + F(1-s)\widetilde{K}(s)}{1-\widetilde{K}(s)\cdot\,\widetilde{K}(1-s)} \qquad (16)$$

This is the operator solution of equation (1). Using the inverse Mellin formula, we find

$$\varphi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{F(s) + F(1-s)\widetilde{K}(s)}{1-\widetilde{K}(s)\,\widetilde{K}(1-s)}x^{-s}\,ds \qquad (17)$$

which is a solution of the integral equation (1).

19.3.3 Example: Solve the integral equation

$$\varphi(x) = f(x) + \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(t)\ \cos xt\, dt \qquad (18)$$

Solution: We have

$$\widetilde{K}(s) = \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty x^{s-1}\ \cos x\, dx \qquad (19)$$

To compute the integral (19), we take advantage of the fact that

$$\int_0^\infty e^{-x}\, x^{z-1}\ dx = \Gamma(z) \qquad (20)$$

If in formula (20) we turn the ray of integration up to the imaginary axis, then using Jordan lemma for $0 < z < 1$, we arrive at the formula

$$\int_0^\infty e^{-ix}\, x^{z-1}\, dx = e^{-\frac{i\pi z}{2}}\,\Gamma(z).$$

Separating the real and imaginary parts, we get

$$\int_0^\infty x^{z-1}\ \cos x\, dx = \cos\frac{\pi z}{2}.\Gamma(z), \qquad (21)$$

$$\int_0^\infty x^{z-1}\ \sin\ x\, dx = \sin\frac{\pi z}{2}.\Gamma(z) \qquad (22)$$

Thus, by virtue of equation (19) and (21), we get

$$\widetilde{K}(s) = \lambda\sqrt{\frac{2}{\pi}}\ \Gamma(s)\cos\frac{\pi s}{2} \qquad (23)$$

Also,

$$\widetilde{K}(s)\cdot\widetilde{K}(1-s) = \lambda\sqrt{\frac{2}{\pi}}\ \Gamma(s)\cos\frac{\pi s}{2}\cdot\lambda\sqrt{\frac{2}{\pi}}\Gamma(1-s)\sin\frac{\pi s}{2}$$

$$= \frac{\lambda^2}{\pi}\ 2\cos\frac{\pi s}{2}\ \sin\frac{\pi s}{2}\ \Gamma(s)\,\Gamma(1-s) = \lambda^2$$

since $\Gamma(s)\cdot\Gamma(1-s) = \frac{\pi}{\sin \pi s}\cdot$ Hence, if $M\{f(x)\} = F(s)$, then by formula (16) (for $|\lambda| \neq 1$)

$$\Phi(s) = \frac{F(s)+F(1-s)\widetilde{K}(s)}{1-\lambda^2}$$

and therefore

$$\varphi(x) = \frac{1}{2\pi i(1-\lambda^2)}\int_{\sigma-i\infty}^{\sigma+i\infty}\left[F(s)+F(1-s)\lambda\sqrt{\frac{2}{\pi}}\,\Gamma(s)\,cos\,\frac{\pi s}{2}\right]x^{-s}\,ds =$$

$$= \frac{1}{1-\lambda^2}\cdot\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)\;x^{-s}\,ds +$$

$$+\;\frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\,cos\,\frac{\pi s}{2}F(1-s)x^{-s}ds \qquad (24)$$

In the second integral on the right of (24) replacing F(1−s) by $\int_0^\infty f(t)t^{-s}\;dt$ and we have $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{F}(s)\,x^{-s}ds\; = f(x)$.

Then formula (24) can be rewritten as

$$\varphi(x) = \frac{f(x)}{1-\lambda^2}+\frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\,cos\,\frac{\pi s}{2}(xt)^{-s}ds\;\int_0^\infty f(t)dt \qquad (25)$$

By Mellin's inversion formula,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\,cos\,\frac{\pi s}{2}(xt)^{-s}ds = cos\;xt$$

So finally, we have

$$\varphi(x) = \frac{f(x)}{1-\lambda^2}+\frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\;\;cos\;xt\;dt,\;\;(|\lambda|\neq 1).$$

19.3.4 Example: Solve the following integral equation

$$\varphi(x) = f(x)+\lambda\sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(t)sinxt\;\;dt$$

Solution: We have $\widetilde{K}(s) = \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty x^{s-1}sinxt\;\;dt = \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty x^{s-1}sinx\;\;dx$ (26)

To compute the integral (26) we take advantage of the fact that

$$\int_0^\infty e^{-x}\,x^{z-1}dx = \Gamma(s) \qquad (27)$$

In equation (27), extending the ray of integration up to the imaginary axis and using Jordan lemma for $0 < z < 1$, we arrive at the formula

$$\int_0^\infty e^{-ix}\,x^{z-1}dx = \;e^{\frac{-i\pi z}{2}}\,\Gamma(s) \qquad (28)$$

Now, separating the real and imaginary parts, we get

$$\int_0^\infty x^{z-1}\,cosx\;\;dx\;\;=\;\;\cos\frac{\pi z}{2}\Gamma(s) \qquad (29)$$

$$\int_0^\infty x^{z-1} \, sinx \, dx \; = sin\frac{\pi z}{2} \;\; \Gamma(s) \qquad (30)$$

Thus, from equations (26) and (30)

$$\widetilde{K}(s) = \lambda\sqrt{\frac{2}{\pi}}\;\Gamma(s)\;\sin\frac{\pi z}{2} \qquad (31)$$

$$\widetilde{K}(s)\widetilde{K}(1-s) = \lambda\sqrt{\frac{2}{\pi}}\Gamma(s)\sin\frac{\pi s}{z}\cdot\lambda\sqrt{\frac{2}{\pi}}\Gamma(1-s)\sin\left(\frac{\pi(1-s)}{2}\right)$$

$$= \frac{\lambda^2 2}{\pi}\Gamma(s)\Gamma(1-s)\sin\frac{\pi s}{2}\sin(90^0 - \frac{\pi s}{2})$$

$$= \frac{\lambda^2 2}{\pi}\sin\frac{\pi s}{2}\;\cos\;\frac{\pi s}{2}\;\Gamma(s)\Gamma(1-s)$$

$$= \frac{\lambda^2}{\pi}\sin\pi s\left(\frac{\pi}{\sin\pi s}\right)$$

$$= \lambda^2$$

If $M\{\,f(x)\} = \;\; F(s)$ then known formula (for $|\lambda| \neq 1$)

$$\Phi(s) = \frac{F(s) + F(1-s)\;\widetilde{K}(s)}{1-\lambda^2}$$

$$\varphi(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left[F(s) + F(1-s)\sqrt{\frac{2}{\pi}}\Gamma(s)\;sin\frac{\pi s}{2}\right]x^{-s}ds$$

$$= \frac{1}{1-\lambda^2}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)x^{-s}\,ds + \frac{\lambda}{1-\lambda^2}\cdot\frac{1}{2\pi i}\sqrt{\frac{2}{\pi}}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)sin\frac{\pi s}{2}\;F(1-s)x^{-s}ds$$

In the second integral replace $F(1-s)$ *by* $\int_0^\infty f(t)t^{-s}dt.$

19.3.5 Note: $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)x^{-s}ds = f(x)$

Then we can write

$$\varphi(x) = \frac{f(x)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)\;sin\frac{\pi s}{2}(xt)^{-s}ds\;\int_0^\infty f(t)dt$$

By Mellin's inversion formula,

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Gamma(s)sin\frac{\pi s}{2}(xt)^{-s}\,ds = \;sinxt$$

$$\varphi(x) = \frac{f(x)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{f}(t)\cos xt\,dt \quad (|\lambda| \neq 1)$$

is the required solution.

19.3.6 Example: Solve the integral equation $\varphi(x) = \frac{1}{1+x^2} + \frac{1}{\sqrt{\pi}}\int_0^\infty \varphi(t)\cos(xt)\ dt.$

Solution: We write the equation in the form

$\varphi(x) = f(x) + \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(t)\cos(xt)\ dt$, where $f(x) = \frac{1}{1+x^2}$.

$\lambda\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{\pi}}$ *then* $\lambda = \frac{1}{\sqrt{2}}$ and the kernel of the integral equation is $cos(xt)$.

The solution of this type of equation is

$$\varphi(x) = \frac{f(x)}{1-\lambda^2} + \frac{\lambda}{1-\lambda^2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(xt)\,dt$$

Substituting $\lambda = \frac{1}{\sqrt{2}}$ *and* $\lambda^2 = \frac{1}{2}$

We obtain the required equation $\varphi(x) = 2f(x) + \sqrt{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(xt)\ dt$

$$\varphi(x) = \frac{2}{1+x^2} + \sqrt{\frac{4}{11}}\int_0^\infty \frac{1}{1+t^2}\cos(xt)\,dt \qquad (32)$$

Solving $\int_0^\infty \frac{1}{1+t^2}\cos(xt)\ dt\ by$ Fourier cosine transforms,

$$F_C(x) = \int_0^\infty f(t)\cos(xt)\ dt$$

$$= \int_0^\infty \frac{\cos(xt)}{1+t^2} = \frac{\pi}{2}e^{-x}, \qquad x > 0$$

$$L\left[\frac{1}{1+t^2}\right] = \int_0^\infty \frac{e^{-st}}{1+t^2}dt = \frac{\pi}{2}e^{-s}, \qquad s > 0$$

$$\int_0^\infty \frac{\cos(xt)}{1+t^2}dt = R\left[\int_0^\infty \frac{e^{ixt}}{1+t^2}\ dt\right]$$

$$s = -ix$$

$$\int_0^\infty \frac{\cos(xt)}{1+t^2}\ dt = \frac{\pi}{2}e^{-x}$$

Putting in equation (32), we obtain

$$\varphi(x) = \frac{2}{1+x^2} + \sqrt{\pi}\,e^{-x}$$

19.4 SUMMARY:

In this section, we discussed two different methods for the solution of certain type of singular integral equations. The first type is based on Efros rule with Laplace transform technique to obtain the solution of singular integral equation. And the second method is based on Mellin transform method. Few examples are discussed for the better understanding of the reader.

19.5 TECHNICAL TERMS:

Efros product rule, Mellin Transform, Fourier transform, Laplace Transform, Gamma Function, Bessel function.

19.6 SELF-ASSESSMENT QUESTIONS:

Solve the following integral equations:

1. $\frac{1}{\sqrt{\pi x}}\int_0^\infty e^{\frac{-t^2}{4x}}\varphi(t)dt = x^{\frac{2}{3}} + e^{4x}.$
2. $\frac{1}{\sqrt{\pi x}}\int_0^\infty e^{\frac{-t^2}{4x}}\varphi(t)dt = 5x - coshx.$
3. $\varphi(x) = -e^{-x} + \frac{2}{\sqrt{\pi}}\int_0^\infty \varphi(t)\cos(xt)\,dt.$
4. $\varphi(x) = 4x + \lambda\sqrt{\frac{2}{\pi}}\int_0^\infty \varphi(t)sinxt\ \ dt.$
5. $\varphi(x) = e^x + \lambda\int_0^\infty \sqrt{\frac{x}{t}}J_1(2\sqrt{xt}\ )\varphi(t)\,dt.$
6. $\varphi(x) = cosx + \lambda\int_0^\infty J_2\left(2\sqrt{xt}\right)\varphi(t)dt.$

Answers to Self-Assessment Questions:

1. $\varphi(x) = \frac{\Gamma\left(\frac{5}{2}\right)}{3!}x^3 + cosh2x$

5. $\varphi(x) = \frac{1}{1-\lambda^2}[e^x - \lambda(e^x - 1)]$

6. $\varphi(x) = \frac{1}{1-\lambda^2}(cosx + \lambda sinx\ )$

19.7 SUGGESTED READINGS:

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LESSON- 20
APPROXIMATE METHODS

OBJECTIVE:

- To learn about three different approximation methods for solving integral equations.
- Replacing the kernel by a degenerate kernel and taking few terms of the Taylor series expansion of the kernel, we approximate the integral. Also, we estimate the error in the approximation.
- Method of successive approximation is used with a sequence of functions as an approximation to the solution.
- Bubnov Galerkin method is used by choosing a system of complete functions which are linearly independent as an approximation to the true solution.

STRUCTURE:

20.1 Introduction

20.2 Replacing the Kernel by a Degenerate Kernel

20.3 Method of Successive Approximation.

20.4 Bubnov-Galerkin method

20.5 Summary

20.6 Technical Terms

20.7 Self-Assessment Questions

20.8 Suggested Readings

20.1 INTRODUCTION:

In this chapter, we will learn about different types of approximation methods use to solve the integral equations. Our focus will be on three different methods. The first method is based on replacing the kernel with a degenerate kernel and using a Taylor series expansion. In the second method, we choose a suitable sequence of functions successively as an approximation to the integral equation. The third approximation method is the Bubnov-Galerkin method, in which a suitable sequence of complete linearly independent functions is selected as an approximation to the integral equation.

20.2 REPLACING THE KERNEL BY A DEGENERATE KERNEL:

Suppose we have an integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\, \varphi(t)\ dt \qquad (1)$$

with an arbitrary kernel $K(x,t)$. The simplicity of finding a solution to an equation with a degenerate kernel led to thinking of replacing the given arbitrary kernel $K(x,t)$ approximately by a degenerate kernel $L(x,t)$ and taking the solution $\tilde{\varphi}(x)$ of the new equation as

$$\tilde{\varphi}(x) = f_1(x) + \lambda \int_a^b L(x,t)\, \tilde{\varphi}(t)\, dt \qquad (2)$$

is also an approximation to the solution of the original equation (1). For the degenerate kernel $L(x,t)$ close to the given kernel $K(x,t)$, we can take a partial sum of Taylor's series for the function $K(x,t)$, a partial sum of the Fourier series for $K(x,t)$, with respect to any complete system of functions $\{u_n(x)\}$ which are orthonormal in $L_2(a,b)$. We shall indicate some error estimates in the solution (1) that occur when replacing a given kernel by a degenerate kernel.

Let there be given two kernels $L(x,t)$ and $K(x,t)$ and let it be known that $\int_a^b |K(x,t) - L(x,t)|\, dt < h$

and that the resolvent kernel $R_L(x,t;\lambda)$ of the equation with the kernel $L(x,t)$ satisfies the inequality

$$\int_a^b |R_L(x,t;\lambda)|\, dt < R$$

and, that $|f(x) - f_1(x)| < \eta$. Then, if the condition $1 - |\lambda| h(1 + |h|R) > 0$, is satisfied, then the equation

$$\varphi(x) = \lambda \int_a^b K(x,t)\varphi(t)\, dt + f(x)$$

has a unique solution $\varphi(x)$ and the difference between this solution and the approximate solution $\tilde{\varphi}(x)$ of the equation

$$\tilde{\varphi}(x) = f_1(x) + \lambda \int_a^b L(x,t)\, \tilde{\varphi}(t)\, dt$$

does not exceed

$$|\varphi(x) - \tilde{\varphi}(x)| < \frac{N|\lambda|(1 + |\lambda|R)^2 h}{1 - |\lambda| h(1 + |\lambda|R)} + \eta \qquad (3)$$

where N is the upper bound of $|f(x)|$.

For the degenerate kernel $L(x,t)$, the resolvent kernel $R_L(x,t;\lambda)$ is found in the evaluation of the integrals. If $L(x,t) = \sum_{k=1}^{n} X_k(x)\, T_k(t)$, then, putting

$$\int_a^b X_k(x)\, T_s(x)\, dx = a_{sk}$$

We get

$$R_L(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} \qquad (4)$$

where

$$D(x,t;\lambda)=\begin{vmatrix} 0 & X_1(t) & \cdots & X_n(x) \\ T_1(t) & 1-\lambda a_{11} & \cdots & -\lambda a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ T_n(t) & -\lambda a_{n1} & \cdots & 1-\lambda a_{nn} \end{vmatrix}, \qquad (5)$$

$$D(\lambda)=\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \cdots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \cdots & -\lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\lambda a_{n1} & -\lambda a_{n2} & \cdots & 1-\lambda a_{nn} \end{vmatrix}, \qquad (6)$$

The roots of $D(\lambda)$ are the characteristic numbers of the kernel $L(x,t)$. Let

$$K(x,t)=L(x,t)+\Lambda(x,t) \qquad (7)$$

where $L(x,t)$ is a degenerate kernel and $\Lambda(x,t)$ has a small norm in some metric. Let $R_k(x,t)$, $R_L(x,t)$ be the resolvent kernels of the kernels $K(x,t)$ and $L(x,t)$, respectively, and $\|\Lambda\|$, $\|R_k\|$, $\|R_L\|$ be the norms of the operators with corresponding kernels. Then

$$\|\varphi(x)-\tilde{\varphi}(x)\| \le \|\Lambda\|\cdot(1+\|R_k\|)\cdot(1+\|R_L\|)\cdot\|f\| \qquad (8)$$

The norm in the above formula (8) can be taken in any function space. The following estimate holds true for the norm of the resolvent kernel R of any kernels $K(x,t)$:

$$\|R\| \le \frac{\|k\|}{1-|\lambda|\cdot\|K\|.} \qquad (9)$$

Let the function space $C(0,1)$ of continuous functions defined on the interval $[0,1]$, then

$$\|K\| := \max_{0\le x\le 1}\int_0^1 |K(x,t)|\,dt$$

$$\|f\| := \max_{0\le x\le 1}|f(x)| \qquad (10)$$

In the space of quadratically summable functions over the domain $\Omega=\{a\le x,t\le b\}$,

$$\|K\| \le \left(\int_a^b\int_a^b K^2(x,t)dx\,dt\right)^{\frac{1}{2}},$$

$$\|f\| \le \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}} \qquad (11)$$

20.2.1 Example: Solve the following equation by replacing its kernel with a degenerate kernel and estimate the error.

$$\varphi(x)=sinx+\int_0^1(1-x\cos xt)\;\varphi(t)\,dt \qquad (1)$$

Solution: Expanding the kernel $K(x,t)=1-x\cos xt$ in Taylor series, we get

$$K(x,t)=1-x+\frac{x^3t^2}{2}-\frac{x^5t^4}{24}+\cdots \qquad (2)$$

Considering the first three terms of the expansion (2) for the degenerate kernel $L(x,t)$,

$$L(x,t) = 1 - x + \frac{x^3t^2}{2} \qquad (3)$$

and solving the new equation

$$\tilde{\varphi}(x) = sinx + \int_0^1 \left(1 - x + \frac{x^3t^2}{2}\right) \tilde{\varphi}(t)\, dt \qquad (4)$$

From equation (4), we have

$$\tilde{\varphi}(x) = sinx + C_1(1-x) + C_2x^3 \qquad (5)$$

where

$$C_1 = \int_0^1 \tilde{\varphi}(t)dt, \qquad C_2 = \frac{1}{2}\int_0^1 t^2\tilde{\varphi}(t)dt \qquad (6)$$

Substituting (5) in (6), we get a system of equations for determining the values of C_1 and C_2. We have

$$C_1 = \int_0^1 [sint + C_1(1-t) + C_2t^3]dt = \frac{1}{2}C_1 + \frac{1}{4}C_2 + 1 - cos\,1,$$

$$C_2 = \frac{1}{2}\int_0^1 [t^2 \sin t + C_1(t^2 - t^3) + C_2t^5]\, dt$$

$$= \frac{1}{24}C_1 + \frac{1}{12}C_2 + \sin 1 - 1 + \frac{1}{2}\cos 1 .$$

or

$$\left.\begin{aligned} \frac{1}{2}C_1 - \frac{1}{4}C_2 &= 1 - cos\,1, \\ \frac{-1}{24}C_1 + \frac{11}{12}C_2 &= sin\,1 + \frac{1}{2}cos\,1 - 1 \end{aligned}\right\} \qquad (7)$$

Solving the above system, we get

$$C_1 = 1.0031, \qquad C_2 = 0.1674$$

and substituting these values in equation (5), we obtain

$$\tilde{\varphi}(x) = 1.0031(1-x) + 0.1674x^3 + sinx .$$

The exact solution of the equation is $\varphi(x) \equiv 1$.

So, let us estimate $\| \varphi - \tilde{\varphi}\|$ using the formula

$$\| \varphi - \tilde{\varphi}\| \le \|\Lambda\| \cdot (1 + \| R_k\|) \cdot (1 + \| R_L\|) \cdot \|f\| \qquad (8)$$

in the metric of the L_2 space. We obtain

$$\|\Lambda\| \le \frac{1}{24}\left\{\int_0^1\int_0^1 x^{10}\, t^8\, dx\, dt\right\}^{\frac{1}{2}} = \frac{1}{72\sqrt{11}} < \frac{1}{238} ,$$

$$\|K\| \le \left\{\int_0^1\int_0^1 [1 - x\cos\, xt]^2 dx\, dt\right\}^{\frac{1}{2}}$$

$$= \left\{2\cos 1 - \tfrac{1}{8}\cos 2 + \tfrac{1}{16}\sin 2 - \tfrac{5}{6}\right\}^{\frac{1}{2}} < \tfrac{3}{5} ,$$

$$\|L\| \le \left\{\int_0^1\int_0^1 [1 - x + \frac{x^3t^2}{2}]^2\, dx\, dt\right\}^{\frac{1}{2}} = \sqrt{\frac{5}{14}} < \frac{3}{5} ,$$

$$\|f\| = \left\{\int_0^1 sin^2(x)dx\right\}^{\frac{1}{2}} = \frac{\sqrt{2 - sin\,2}}{2} < \frac{3}{5} .$$

Finally, we estimate the norms of the resolvent kernels R_k and R_L using the formulas as

$$\|R_k\| \leq \frac{\|K\|}{1-|\lambda|\cdot\|K\|}, \quad \|R_L\| \leq \frac{\|L\|}{1-|\lambda|\cdot\|L\|}$$

where $|\lambda| = 1$. Hence, $\|R_k\| \leq \frac{3}{2}, \quad \|R_L\| \leq \frac{3}{2}$ and

$$\| \varphi - \tilde{\varphi}\| < \frac{1}{238}\left(1+\frac{3}{2}\right)\left(1+\frac{3}{2}\right)\cdot\frac{3}{5} < 0.016.$$

20.2.2 Example: Find the solution of the integral equation by substituting a degenerate kernel and estimate the error.

$$\varphi(x) = e^x - x - \int_0^1 x(e^{xt}-1)\varphi(t)dt$$

Solution: Given that $\varphi(x) = e^x - x - \int_0^1 x(e^{xt}-1)\varphi(t)dt$. We aim to approximate the kernel $K(x,t) = x(e^{xt}-1)$ with a degenerate kernel $K(x,t) = \sum_{n=1}^{N} a_n(x)b_n(t)$

Let $e^{xt} = 1 + xt + \frac{(xt)^2}{2!} + \frac{(xt)^3}{3!} + \cdots$

$$e^{xt} - 1 = xt + \frac{(xt)^2}{2} + \frac{(xt)^3}{6} + \frac{(xt)^4}{24}$$

$$K(x,t) = x(e^{xt}-t) = x^2t + \frac{x^3t^2}{2} + \frac{x^4t^3}{6} + \cdots$$

Keep up to 4^{th} order $K(x,t) = x^2t + \frac{x^3t^2}{2} + \frac{x^4t^3}{6} + \frac{x^5t^4}{24}$ ...

Then $\varphi(x) = e^x - x - \int_0^1\left(x^2t + \frac{x^3t^2}{2} + \frac{x^4t^3}{6}\right)\varphi(t)dt$

$$= e^x - x - x^2\int_0^1 t\,\varphi(t)dt - \int_0^1 \frac{x^3t^2}{2}\varphi(t)\,dt - \int_0^1 \frac{x^4t^3}{6}\varphi(t)dt$$

$$= e^x - x - x^2\int_0^1 t\varphi(t)dt - \frac{x^3}{2}\int_0^1 t^2\,\varphi(t)dt - \frac{x^4}{6}\int_0^1 t^3\,\varphi(t)dt$$

$\varphi(t) \simeq 1$ for first iteration

$$\int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\int_0^1 t^3 dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\varphi(x) = e^x - x - \frac{x^2}{2} - \frac{1}{3}\frac{x^3}{2} - \frac{x^4}{24}$$

$$\varphi(x) = e^x - x - 0.5x^2 - 0.1667x^3 - 0.0417x^4$$

Error Estimate Formula:

$$||\varphi - \tilde{\varphi}|| \leq \ || \Lambda || \cdot (1 + \ ||R_k||)\ (\ 1 + ||R_L||) \cdot ||f||$$

$$|| \Lambda || \leq \ \frac{1}{24}\int_0^1 \int_0^1 ((x^5 t^4 \ )^2)^{\frac{1}{2}}\ dx\ dt$$

$$\leq \ \frac{1}{24}\left(\int_0^1 \int_0^1 x^{10}\ t^8 dx\ dt\right)^{\frac{1}{2}}$$

$$\leq \ \frac{1}{24}\sqrt{\int_0^1 x^{10}\ dx \cdot \ \int_0^1 t^8 dt}$$

$$\leq \ \frac{1}{24}\sqrt{\frac{1}{99}}$$

$$\leq \ \frac{1}{24} \cdot \frac{1}{9.95} \simeq \frac{1}{238}$$

Kernel: $K(x,t) = x(e^{xt} - 1)$

$$||K|| = \ \left(\int_0^1 \int_0^1 [x(e^{xt} - 1)]^2\ dx\ dt\ \right)^{\frac{1}{2}}$$

$$= \ \left(\int_0^1 \int_0^1 \left(x^2 t + \frac{x^3 t^2}{2}\right)^2\ dx\ dt\ \right)^{\frac{1}{2}}$$

$$= \ \left(\int_0^1 \int_0^1 \left(x^4 t^2 + \frac{x^6 t^4}{4} + x^5 t^3\right) dx\ dt\ \right)^{\frac{1}{2}}$$

We obtain $\int_0^1 \int_0^1 x^4 t^2\ dt\ dx = \ \int_0^1 x^4 dx \cdot \int_0^1 t^2 dt = \ \frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$.

$$\int_0^1 \int_0^1 x^5 t^3\ dt\ dx = \frac{1}{24}$$

$$\int_0^1 \int_0^1 \frac{x^6 t^4}{4} dt\ dx = \ \frac{1}{140}$$

So, $|| K|| \simeq \ \left(\frac{1}{15} + \frac{1}{24} + \frac{1}{140}\right)^{\frac{1}{2}} = \ \left(\frac{97}{840}\right)^{\frac{1}{2}} = 0.340 < \frac{3}{5}$

Kernel $L(x,t)$:

$$L(x,t) = \ \left(x^2 t + \frac{x^3 t^2}{2} + \frac{x^4 t^3}{6}\right)$$

$$||L|| = \ \left(\int_0^1 \int_0^1 \left(x^2 + \frac{x^3 t^2}{2} + \frac{x^4 t^3}{6}\right)^2\ dx\ dt\right)^{\frac{1}{2}}$$

We obtain,

$$\int_0^1 \int_0^1 x^4 t^2\ dx\ dt = \ \frac{1}{15}$$

$$\int_0^1 \int_0^1 \frac{x^6 t^4}{4}\ dx\ dt = \ \frac{1}{140}$$

$$\int_0^1 \int_0^1 \frac{x^8 t^6}{36}\ dx\ dt = \ \frac{1}{2268}$$

$$|| L||^2 = \ \frac{1}{15} + \frac{1}{140} + \frac{1}{2268} = \ \frac{842}{2268}$$

$$||L|| \simeq \sqrt{0.07425} = 0.272 < \frac{3}{5}$$

We have $f(x) = \ e^x - x$.

$$||f|| = \left(\int_0^1 (e^x - x)^2\ dx\ \right)^{\frac{1}{2}}$$

Now $(e^x - x)^2 = \left(1 + \frac{x^2}{2} + \frac{x^3}{6} \ldots\right)^2 \simeq 1 + \frac{x^4}{4}$

$||f|| = \left(\int_0^1\left(1+\frac{x^4}{4}\right)dx\right)^{\frac{1}{2}} = \frac{21}{20} < \frac{4}{3}$

$||\, R_k|| \le \frac{||K||}{1-|\lambda|\cdot||\,K||}$, with $|\lambda| = 1, we\ obtain\ \ ||\,R_k|| = 0.5\ < \frac{3}{2}$

$||\, R_L|| \le \frac{||L||}{1-|\lambda|\cdot||\,L||}$, with $|\lambda| = 1\ then\ ||\,R_L|| = 0.3736 < \frac{3}{2}$

Hence, we have the error estimate as

$||\varphi - \tilde{\varphi}|| < \frac{1}{238}(1+0.5)(1+0.3736)\cdot 1.05 = 0.009 < 0.16$.

20.2.3 Example: Find the solution of the Integral equation by substituting a degenerate kernel K and estimate the error.

$$\varphi(x) = \frac{1}{2}(e^{-x} + 3x - 1) + \int_0^1 (e^{-xt^2} - 1)x\,dt$$

Solution: Given that $\varphi(x) = \frac{1}{2}(e^{-x} + 3x - 1) + \int_0^1 (e^{-xt^2} - 1)x\,dt$ (1)

where $e^{-xt^2} = 1 - xt^2 + \frac{x^2t^4}{2} - \frac{x^3t^6}{3!} + \frac{x^4t^8}{4!} \ldots$

$e^{-xt^2} - 1 = -xt^2 + \frac{x^2t^4}{2} - \frac{x^3t^6}{3!} + \frac{x^4t^8}{4!} \ldots$

$\left(e^{-xt^2} - 1\right)x = -x^2t^2 + \frac{x^3t^4}{2} - \frac{x^4t^6}{3!} + \frac{x^5t^8}{4!} + \cdots$ (2)

Substituting equation (2) in (1), we have

$$\varphi(x) = \frac{1}{2}(e^{-x} + 3x - 1) + \int_0^1 \left(-x^2t^2 + \frac{x^3t^4}{2} - \frac{x^4t^6}{3!}\right)\varphi(t)dt$$

Using approximate estimate $\varphi(x) \simeq 1$.

We have, $\int_0^1 -x^2\,t^2\varphi(t)dt = -x^2\int_0^1 t^2\ dt = -\frac{x^2}{3}$

$\int_0^1 \frac{x^3t^4}{2}\varphi(t)dt = \frac{x^3}{2}\int_0^1 t^4\ dt = \frac{x^3}{10}$

$$-\int_0^1 \frac{x^4t^6}{3!}\varphi(t)dt = -\frac{x^4}{6}\int_0^1 t^6\,dt = -\frac{x^4}{42}$$

$$\varphi(x) = \frac{1}{2}(e^{-x} + 3x - 1) - 0.333x^2 + 0.1x^3 - 0.0238x^4$$

Using the error estimate Formula,

$||\varphi - \tilde{\varphi}|| \le ||\,\Lambda\,|| \cdot (1 + ||R_k||)\,(1 + ||R_L||) \cdot ||f||$

$$||\,\Lambda\,|| \le \frac{1}{24}||\,x^5t^8|| = \frac{1}{24}\left(\int_0^1\int_0^1 x^{10}t^{16}\right)^{\frac{1}{2}} dx\,dt$$

$$= \frac{1}{24}\sqrt{\frac{1}{11}\cdot\frac{1}{17}} = \frac{1}{24}\sqrt{\frac{1}{187}} = 0.0030$$

Kernel $K(x,t) = \left(e^{-xt^2} - 1\right)x$

$$||\,K|| = \left(\int_0^1\int_0^1 \left(xe^{-xt^2} - x\right)^2 dx\,dt\right)^{\frac{1}{2}}$$

$||K|| = \left(\int_0^1\int_0^1 (x^4t^4 + \frac{x^6t^8}{4} - \frac{2x^5t^6}{2}\ dx\,dt\right)^{\frac{1}{2}}$

$||K|| = \left(\frac{1}{25} + \frac{1}{1008} - \frac{1}{66}\right)^{\frac{1}{2}}$

$||K|| = (0.0258)^{\frac{1}{2}} = 0.1606 < \frac{3}{5}$.

Degenerate Kernel $L(x,t)$:

$$L(x,t) = -x^2t^2 + \frac{x^3t^4}{2!} - \frac{x^4t^6}{3!}$$

$$|| L|| = \left(\int_0^1 \int_0^1 \left(-x^2t^2 + \frac{x^3t^4}{2} - \frac{x^4t^6}{3!}\right)^2 dx\ dt\right)^{\frac{1}{2}}$$
$$= \left(\int_0^1 \int_0^1 \left(x^4t^4 + \frac{x^6t^8}{4} + \frac{x^8t^{12}}{36}\right) dx\ dt\right)^{\frac{1}{2}}$$
$$= (0.044849)^{\frac{1}{2}}$$
$$= 0.211775 < \frac{1}{2}$$

$$f = \frac{1}{2}(e^{-x} + 3x - 1)$$
$$|| f|| = \int_0^1 \int_0^1 \frac{1}{2}(e^{-x} + 3x - 1)^2\ dx$$

We know that $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots$

So, $e^{-x} + 3x - 1 = 2x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots$

$$f = 2x^2 + \frac{x^2}{2} - \frac{x^3}{3!}$$
$$|| f|| = \left(\int_0^1 (e^{-x} + 3x - 1)^2\right)^{\frac{1}{2}}$$
$$= \left(1 + \frac{1}{6} - \frac{1}{24} + \frac{1}{120}\right)^{\frac{1}{2}}$$
$$= \left(\frac{134}{120}\right)^{\frac{1}{2}}$$
$$= (1.1166)^{\frac{1}{2}} = 1.0566 < \frac{4}{3}.$$

$|| R_k|| \leq \frac{||K||}{1-|\lambda|\cdot|| K||}$ where $|\lambda| = 1\ then\ || R_k|| = 0.19132 < \frac{3}{2}$

$|| R_L|| \leq \frac{||L||}{1-|\lambda|\cdot|| L||}$ where $|\lambda| = 1\ then\ || R_L|| = -0.1758 < \frac{3}{2}$

$$||\varphi - \tilde{\varphi}|| \leq 0.0030(1 + 0.1913)(1 - 0.1758) \cdot (1.0566)$$
$$\leq (0.00357)(0.87084) \leq 0.003.$$

20.3 METHOD OF SUCCESSIVE APPROXIMATION:

Considering an integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)dt \qquad (1)$$

We construct a sequence of functions $\{\varphi_n(x)\}$ with the recursion formula

$$\varphi_n(x) = f(x) + \lambda \int_a^b K(x,t)\ \varphi_{n-1}(t)dt \qquad (2)$$

The functions $\varphi_n(x)$, $(n = 1,2, \dots)$ are considered as approximations to the desired solution of the equation (1). The zero approximation $\varphi_0(x)$ may be chosen arbitrarily.

Under certain conditions

$$|\lambda| < \frac{1}{B}, \qquad B = \sqrt{\int_a^b \int_a^b K^2(x,t)dx\ dt} \qquad (3)$$

The sequence in equation (2) converges to the solution of equation (1). The magnitude of the error of the $(m + 1)th$ approximation is given by the inequality

$$|\varphi(x) - \varphi_{m+1}(x)| \leq FC_1B^{-1}\frac{|\lambda B|^{m+1}}{1-|\lambda B|} + \Phi C_1 B^{-1}|\lambda B|^{m+1} \qquad (4)$$

where

$$F = \sqrt{\int_a^b f^2(x)dx}\ , \qquad \Phi = \sqrt{\int_a^b \varphi_0^2\ (x)dx}\ ,$$

$$C_1 = \sqrt{\max_{a\leq x\leq b}\int_a^b K^2(x,t)\ dt}$$

20.3.1 Observation: The basic difficulty in applying the method of successive approximations consists in computing the integrals as given in formula (2). As a rule, it is performed with the formulas of approximate integration. Therefore, it is advisable to replace the given kernel by a degenerate kernel with the help of a Taylor expansion and then introduce the iteration method.

20.3.2 Example: Solve the following problem using the method of successive approximation $\varphi(x) = 1 + \int_0^1 xt^2\varphi(t)dt$.

Solution: Given that $\varphi(x) = 1 + \int_0^1 xt^2\varphi(t)dt$

This is a linear Fredholm integral equation of the second kind with a separable kernel $K(x,t) = xt^2$.

Step 1: initial approximation: $\varphi_0(x) = 1$.

Step 2: Recursive Formula: $\varphi_{n+1}(x) = 1 + x\int_0^1 t^2\ \varphi_n(t)dt$.

Iteration 1: $\varphi_1(x) = 1 + x\int_0^1 t^2 \cdot 1\ dt = 1 + x\left[\frac{t^3}{3}\right]_0^1 = 1 + \frac{x}{3} = 1 + 0.3x.$

Iteration 2: $\varphi_2(x) = 1 + x\int_0^1 t^2\left(1+\frac{t}{3}\right)dt$

$$= 1 + x\int_0^1\left(t^2 + \frac{t^3}{3}\right)dt$$

$$= 1 + x\left[\frac{1}{3} + \frac{1}{12}\right]$$

$$= 1 + \frac{5x}{12} = 1 + 0.416x.$$

Iteration 3: $\varphi_3(x) = 1 + x\int_0^1 t^2\left(1+\frac{5t}{12}\right)dt$

$$= 1 + x\left[\frac{t^3}{3} + \frac{5t^4}{48}\right]_0^1$$

$$= 1 + x\left[\frac{16}{48} + \frac{5}{48}\right]$$

$$= 1 + x\left[\frac{21}{48}\right] = 1 + 0.4375x$$

This seems to convergence to $\varphi(x) = 1 + \frac{4}{9}x$.

So,

$$\int_0^1 xt^2\left(1+\frac{4t}{9}\right)dt = x\left(\frac{1}{3}+\frac{4}{9}\cdot\frac{1}{4}\right) = x\left(\frac{1}{3}+\frac{1}{9}\right) = x\cdot\frac{4}{9}$$

$$\varphi(x) = 1+\frac{4x}{9}.$$

20.3.3 Example: Solve the following equations using the method of successive approximations

$$\varphi(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 xt\varphi(t)dt$$

Solution: Given that $\varphi(x) = \frac{5}{6}x + \frac{1}{2}\int_0^1 xt\varphi(t)dt$, the separable kernel $K(x,t) = xt$.

Iteration Approximation: $\varphi_0(x) = 0$.

Recursive formula: $\varphi_{n+1}(x) = \frac{5x}{6} + \frac{1}{2}x\int_0^1 t\varphi_n(t)dt$.

Iteration 1: $\varphi_{0+1}(x) = \frac{5x}{6} + \frac{1}{2}x\int_0^1 t\cdot 0\, dt = \frac{5}{6}x$.

Iteration 2: $\varphi_2(x) = \frac{5}{6}x + \frac{1}{2}x\int_0^1 t\cdot\frac{5}{6}t\, dt$

$$= \frac{5}{6}x + \frac{1}{2}x\left[\frac{5}{6}.\frac{1}{3}\right]$$

$$= \frac{5}{6}x + \frac{5}{36}x$$

$$= \frac{35}{36}x.$$

Iteration 3: $\varphi_3(x) = \frac{5}{6}x + \frac{1}{2}x\int_0^1 \frac{35}{36}t^2 dt$

$$= \frac{5}{6}x + \frac{1}{2}x\frac{35}{108}$$

$$= \frac{215}{216}x.$$

Iteration 4: $\varphi_4(x) = \frac{5}{6}x + \frac{1}{2}x\int_0^1 \frac{215}{216}t^2\, dt$

$$= \frac{5}{6}x + \frac{1}{2}x\frac{215}{648}$$

$$= \frac{129.6(5)+2.15(6)}{1296(6)}\, x$$

$$= \frac{7770}{7776}x$$

$$\therefore\quad \varphi(x) = x$$

Verification:

$$\varphi(x) = \frac{5}{6}x + \frac{1}{2}x\int_0^1 t\cdot t\, dt$$

$$\varphi(x) = \frac{5}{6}x + \frac{1}{2}x\int_0^1 t^2\, dt$$

$= \frac{5}{6}x + \frac{1}{2}x\left[\frac{t^3}{3}\right]_0^1$

$= \frac{5}{6}x + \frac{1}{2}x\left[\frac{1}{3}\right] = x$

$\varphi(x) = x.$

20.4 BUBNOV-GALERKIN METHOD:

An approximate solution of the integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\ \varphi(t)\, dt \qquad (1)$$

by means of the Bubnov- Galerkin method is sought in the following manner. First, we choose a system of functions $\{u_n(x)\}$, which is complete in $L_2(a,b)$. For any n the sequence of functions $u_1(x), u_2(x), \dots, u_n(x)$ are linearly independent and we seek the approximate solution $\varphi_n(x)$ in the form

$$\varphi_n(x) = \sum_{k=1}^{n} a_k\, u_k(x) \qquad (2)$$

The coefficients $a_k (k = 1,2,\dots,n)$ are found from the following linear system:

$$(\varphi_n(x),\ u_k(x)) = \big(f(x)\ u_k(x)\big) + \lambda\left(\int_a^b K(x,t)\ \varphi_n(t)\, dt,\ u_k(x)\right)$$

$$(k = 1,2,\dots\dots,n) \qquad (3)$$

Where the inner product (f,g) stands for $\int_a^b f(x)g(x)dx$ and in place of $\varphi_n(x)$ we have to substitute $\sum_{k=1}^{n} a_k\, u_k(x)$. If the value of λ in (1) is not a characteristic value, then the system in (3) is uniquely solvable for sufficiently large values of n, as $n \longrightarrow \infty$. The approximate solution $\varphi_n(x)$ tends to the exact solution $\varphi(x)$ in $L_2(a,b)$.

20.4.1 Example: Use the Bubnov-Galerkin method to solve the equation

$$\varphi(x) = x + \int_{-1}^{1} xt\, \varphi(t) dt \qquad (4)$$

Solution: Here, for a complete system of functions on $[-1,1]$, we choose the system of Legendre polynomials $P_n(x)$ $(n = 0,1,2,\dots)$. We look for the approximate solution $\varphi_n(x)$ of equation (4) in the form

$$\varphi_3(x) = a_1 \cdot 1 + a_2 x + a_3\left(\frac{3x^2-1}{2}\right)$$

Substituting $\varphi_3(x)$ in place of $\varphi(x)$ in equation (4), we get

$$a_1 + a_2 x + a_3(\frac{3x^2-1}{2}) = x + \int_{-1}^{1} xt\left(a_1 + a_2 t + a_3\frac{3t^2-1}{2}\right)dt$$

or

$$a_1 + a_2 x + a_3\left(\frac{3x^2 - 1}{2}\right) = x + x\frac{2}{3}a_2 \qquad (5)$$

Multiplying both sides of equation (5) successively by $1, x, \frac{3x^2-1}{2}$ respectively, and integrating with respect to x between the limits -1 and 1. We obtain

$$2a_1 = 0,$$

$$\frac{2}{3}a_2 = \frac{2}{3} + \frac{4}{9}a_2,$$

$$\frac{2}{5}a_3 = 0$$

When $a_1 = 0, a_2 = 3, a_3 = 0$, then $\varphi_3(x) = 3x$, It is easy to verify that this is the exact solution of equation (4).

20.4.2 Note: Bubnov - Galerkin method yields an exact solution for degenerate kernels; for the general case, it is equivalent to replacing the kernel $K(x, t)$ by the degenerate kernel $L(x, t)$.

20.4.3 Example: Solve the following integral equations by using Bubnov Galerkin method

$$\varphi(x) = 1 + \int_{-1}^{1}(xt + x^2)\varphi(t)dt$$

Solution: For the complete system of functions on $[-1, 1]$, we choose the system of Legendre polynomials $p_n(x)$ $(n = 0, 1, 2, 3 \dots \dots)$. We seek the approximate solution $\varphi_n(x)$ of the equation in the form

Rodriques formula: $p_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n$

Derivation of $p_n(x)$:

Let $n = 0$.

$$p_0(x) = \frac{1}{2^0\, 0!} \cdot \frac{d^0}{dx^0}(x^2 - 1)^0 = 1$$

Derivation of $p_1(x)$:

Let $n = 1$

$$p_1(x) = \frac{1}{2^1\, 1!} \cdot \frac{d}{dx}(x^2 - 1)$$

$$p_1(x) = \frac{1}{2}\frac{d}{dx}(x^2 - 1)$$

$$p_1(x) = x$$

Derivation of $p_2(x)$:

Let $n = 2$

$p_2(x) = \frac{1}{2^2 2!} \cdot \frac{d^2}{dx^2}(x^2-1)^2$

$p_2(x) = \frac{1}{8}(4x^3 - 4x)^1$

$p_2(x) = \frac{1}{8}(12x^2 - 4)$

$p_2(x) = \frac{3x^2-1}{2}.$

$$\varphi_3(x) = a_1 + a_2(x) + a_3\left(\frac{3x^2-1}{2}\right)$$

$$\varphi(x) = 1 + \int_{-1}^{1}(xt + x^2)\varphi(t)dt$$

$$\varphi(x) = 1 + x\int_{-1}^{1} t\varphi(t)\, dt \;+ x^2\int_{-1}^{1}\varphi(t)dt$$

$$\varphi(x) = 1 + xC_1 + x^2C_2$$

We know that $\varphi(t) = a_1 + a_2(t) + a_3\left(\frac{3t^2-1}{2}\right)$

$C_1 = \int_{-1}^{1} t\varphi(t)dt$

$= \int_{-1}^{1} t\left(a_1 + a_2 t + a_3\left(\frac{3t^2-1}{2}\right)\right) dt$

$= \int_{-1}^{1} a_1(t)dt + \int_{-1}^{1} a_2\, t^2 dt + \int_{-1}^{1} a_3\left(\frac{3t^2-1}{2}\right) dt$

We know that an odd function is zero.

$C_1 = \frac{2}{3}a_2$

$C_2 = \int_{-1}^{1}\varphi(t)dt = \int_{-1}^{1}\left(a_1 + a_2 t + a_3\left(\frac{3t^2-1}{2}\right)\right)dt = 2a_1$

$\therefore\ \varphi(x) = 1 + \frac{2}{3}a_2 x + x^2 2a_1$ (1)

$$\varphi_3(x) = a_1 + a_2 x + \frac{a_3\, 3x^2}{2} - \frac{a_3}{2} = \left(a_1 - \frac{a_3}{2}\right) + a_2 x + \frac{3a_3}{2}x^2 \qquad (2)$$

Comparing equations (1) and (2), we get $a_1 = 3,\ a_2 = 0,\ a_3 = 4.$

Substituting these values, we get $\varphi(x) = 6x^2 + 1.$

20.4.4 Example: Solve the integral equation by using the Bubnov-Galerkin method

$$\varphi(x) = 1 + \frac{4}{3}x + \int_{-1}^{1} (xt^2 - x)\varphi(t)\, dt$$

Solution: Using a 3-term Galerkin expansion in Legendre polynomials

$\varphi(x) = \ a_1 + a_2(x) + a_3(\frac{3x^2-1}{2})$

We write $\varphi(x) = 1 + \frac{4}{3}x + x\int_{-1}^{1}(t^2 - 1)\varphi(t)\, dt$

$= 1 + \frac{4}{3}x + xc$

$$C = \int_{-1}^{1} (t^2 - 1)\, \varphi(t)\, dt$$

$\varphi(t) = \ a_1 + a_2(t) + a_3(\frac{3t^2-1}{2})$

$= \ \int_{-1}^{1}(t^2 - 1)\left(a_1 + a_2 t + a_3\left(\frac{3t^2-1}{2}\right)\right) dt$

$= \int_{-1}^{1} a_1\,(t^2 - 1) + a_2(t^3 - t) + a_3\left(\frac{3t^4-4t^2+1}{2}\right) dt$

$= -\frac{4}{3}a_1 + \frac{a_3}{2}\frac{8}{15}$

$\varphi(x) = 1 + \left[\frac{4}{3} - \frac{4}{3}a_1 + \frac{4a_3}{15}\right] x$

Now $\varphi(x) = \ a_1 + a_2 x + \frac{a_3}{2}(3x^2 - 1) = \ \left(a_1 - \ \frac{a_3}{2}\right) + a_2 x + \frac{3a_3}{2}\ x^2$

Comparing the coefficients of x, x^2, constants, we get $a_1 = 1,\ \ a_2 = 0, a_3 = 0$

Now we get $\varphi(x) = 1$

Hence $\varphi(x) = 1$ is the exact solution.

20.5 SUMMARY:

In this section, we learnt about three different approximation methods for solving integral equations. In the first approximation method, we replace the kernel by a degenerate kernel and take a few terms of the Taylor series expansion of the kernel, we approximate the integral. Also, we estimate the error of the exact to the approximate solution in a suitable norm. The second kind of method is successive approximation, where we use a sequence of functions as an approximation to the solution. The third type of method is Bubnov Galerkin method, which is based on choosing a system of complete functions which are linearly independent as an approximation to the true solution. Few examples in each category have been discussed for the better understanding of the reader.

20.6 TECHNICAL TERMS:

Taylor series, Bubnov-Galerkin Method, degenerate kernel

20.7 SELF-ASSESSMENT QUESTIONS:

1. Solve the following equations by replacing its kernel with a degenerate kernel and estimate the error.

$$(i)\ \varphi(x) = x + cosx + \int_0^1 x(\sin xt - 1)\ \varphi(t)\ dt.$$

$$(ii)\ \ \varphi(x) = \frac{x}{2} + \frac{\sin x}{2} + \int_0^1 (1 - \cos x\, t^2)\, x\, \varphi(t)\ dt.$$

2. Solve the following equations using the method of successive approximations.

$$(i)\varphi(x) = 3 + \int_0^1 xt^3\varphi(\,t)dt.$$

$$(ii)\varphi(x) = 5x + \frac{7}{4}\int_0^1 xt\varphi(\,t)dt.$$

3. Solve the following integral equations by Bubnov- Galerkin method

$$(i)\ \varphi(x) = 1 - x(e^x - e^{-x}) + \int_{-1}^1 x^2 e^{xt}\varphi(t)\ dt.$$

$$(ii)\ \ \varphi(x) = 3x + \int_{-1}^1 (xt - x^2)\varphi(t)dt.$$

Answers to Self-Assessment Questions:

1. (i) $\tilde{\varphi}(x) = cosx + \frac{x}{89}[78 - 78\, sin1 - 24cos1 + x(84\sin 1 + 108cos1 - 84)]$;
   $|\varphi - \tilde{\varphi}| < 0.040$; The exact solution is $\varphi(x) \equiv 1$
   (ii) $\tilde{\varphi}(x) = \frac{x}{2} + \frac{1}{2}\sin x + \left(\frac{58}{9} - \frac{16}{3} sin1 - \frac{52}{15} cos1\right) x^3$; $|\varphi - \tilde{\varphi}| < 0.0057$;
   The exact solution is $\varphi(x) = x$.
2. $(i)\ \varphi(x) = 3 + \frac{3}{4}x$.

3. $(i)\ \varphi_3(x) =\ 1$ is the exact solution.

20.8 SUGGESTED READINGS:

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