ANALYSIS -II

M.Sc., MATHEMATICS First Year

Semester – II, Paper-II

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M.Sc., MATHEMATICS – ANALSIS -II

First Edition 2025

No. of Copies :

Published by:

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Director, I/C
Centre for Distance Education,
Acharya Nagarjuna University

Printed at:

FOREWORD

Since its establishment in 1976, Acharya Nagarjuna University has been forging ahead in the path of progress and dynamism, offering a variety of courses and research contributions. I am extremely happy that by gaining 'A$^+$' grade from the NAAC in the year 2024, Acharya Nagarjuna University is offering educational opportunities at the UG, PG levels apart from research degrees to students from over 221 affiliated colleges spread over the two districts of Guntur and Prakasam.

The University has also started the Centre for Distance Education in 2003-04 with the aim of taking higher education to the doorstep of all the sectors of the society. The centre will be a great help to those who cannot join in colleges, those who cannot afford the exorbitant fees as regular students, and even to housewives desirous of pursuing higher studies. Acharya Nagarjuna University has started offering B.Sc., B.A., B.B.A., and B.Com courses at the Degree level and M.A., M.Com., M.Sc., M.B.A., and L.L.M., courses at the PG level from the academic year 2003-2004 onwards.

To facilitate easier understanding by students studying through the distance mode, these self-instruction materials have been prepared by eminent and experienced teachers. The lessons have been drafted with great care and expertise in the stipulated time by these teachers. Constructive ideas and scholarly suggestions are welcome from students and teachers involved respectively. Such ideas will be incorporated for the greater efficacy of this distance mode of education. For clarification of doubts and feedback, weekly classes and contact classes will be arranged at the UG and PG levels respectively.

It is my aim that students getting higher education through the Centre for Distance Education should improve their qualification, have better employment opportunities and in turn be part of country's progress. It is my fond desire that in the years to come, the Centre for Distance Education will go from strength to strength in the form of new courses and by catering to larger number of people. My congratulations to all the Directors, Academic Coordinators, Editors and Lesson-writers of the Centre who have helped in these endeavors.

Prof. K. Gangadhara Rao
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M.Sc. – MATHEMATICS SYLLABUS

SEMESTER – II

202MA24 :: ANALYSIS – II

UNIT-I: Sequences and series of functions: Discussion of main problem, Uniform convergence, Uniform convergence and Continuity, Uniform convergence and Integration. (7.1 to 7.16 of Chapter 7 of the Text Book)

UNIT-II: Uniform Convergence and Differentiation, Equicontinuous families of functions, Stone-Weierstrass theorem. (7.17 to 7.27 of Chapter 7 of the Text Book)

UNIT-III: Algebra of functions, Power series, Exponential and logarithmic functions, Trigonometric functions. (7.28 to 7.33 of Chapter 7 and 8.1 to 8.7 of Chapter 8 of the Text Book)

UNIT-IV: Linear transformations, Differentiation, Contraction principle, Inverse function theorem. (9.1 to 9.25 of Chapter 9 of the Text Book)

UNIT-V: Implicit function theorem, Determinants, Derivatives of higher order, Differentiation of integrals.
(9.26 to 9.29 and 9.33 to 9.43 of Chapter 9 of the Text Book)

TEXT BOOK: Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.

REFEREBCE BOOK:

1. Mathematical Analysis by Tom M. Apostal, Narosa Publishing House, 2nd Edition,1985.
2. Mathematical Analysis by S.C. Malik and Savita Arora, New Age International (P) Limited, 2nd Edition, 1997.

CODE: 202MA24

M.Sc DEGREE EXAMINATION
Second Semester
Mathematics:: Paper II - ANALYSIS-1I

MODEL QUESTION PAPER

Time : Three hours **Maximum : 70 marks**

Answer ONE question from each Unit. **(5 x 14 = 70)**

UNIT-I

1. State and prove Weierstross M-Test for uniform convergence theorem.

or

2. State and prove uniform convergence and integration theorem.

UNIT-II

3. Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \le x \le b)$.

or

4. (a) If $\{f_n\}$ is a point wise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a sub sequence $\{f_{nk}\}$ such that $\{f_{nk}(x)\}$ converges for every $x \in E$.

   (b) If K is compact metric space. If $f_n \in \mathbb{C}(K)$ for n = 1, 2, 3,…. and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is Equicontinuous on K.

UNIT-III

5. Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set K. If $\mathcal{A}$ seperates points on K and if $\mathcal{A}$ vanishes at no point of K, then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all real continuous functions on K.

or

6. Suppose $\sum c_n$ converges. Put $f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad (-1 < x < 1)$.
   Then prove that $\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} c_n$.

UNIT-IV

7. (a) Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then dim$X \leq r$.

   (b) A linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

or

8. Suppose $\bar{f}$ maps a convex open set $E \subset R^n$ into R^m, $\bar{f}$ is differentiable in E, and there is a real number M such that $\|\bar{f}'(x)\| \leq M$ for every $\bar{x} \in E$. Then $|\bar{f}(\bar{b}) - \bar{f}(\bar{a})| \leq M|\bar{b} - \bar{a}|$ for all $\bar{a} \in E, \bar{b} \in E$.

UNIT-V

9. a) Prove that a linear operator A on R^n is invertible if and only if $\det[A] \neq 0$.

   b) Suppose f is defined in an open set $E \subset R^2$. Suppose that $D_1 f, D_{21} f$ and $D_2 f$ exist at every point of E, and $D_{21} f$ is continuous at some point $(a, b) \in E$.

or

10. If $[A]$ and $[B]$ are n by n matrices, then prove that $\det([B][A]) = \det[B] \det[A]$.

CONTENTS

LESSON - 1

SEQUENCES AND SERIES OF FUNCTIONS: DISCUSSION AND MAIN PROBLEM AND UNIFORM CONVERGENCE

OBJECTIVE:

After studying the lesson you should able to understand the concept of point wise convergence and uniform convergence of functions.

STRUCTURE:

1.1 Introduction
1.2 Definition of point wise convergence
1.3 Lemma and examples
1.4 Uniform convergence and related theorems
1.5 Summary
1.6 Technical terms
1.7 Self -Assessment Questions
1.8 Suggested readings

1.1 INTRODUCTION:

In this lesson, we define and study the convergence of sequences and series of functions. There are many different ways to define the convergence of a sequence of functions, and different definitions lead to in equivalent types of convergence. We consider here two basic types: point wise and uniform convergence.

1.2 DEFINITION:

Let E be a set, $\{f_n\}, n = 1,2,3 ... \infty$ sequence of functions defined on E and let f be a function defined on E.

i. We say that the sequence $\{f_n\}$ converges to f pointwise or converges pointwise to f on E if for every $x \in E$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$ (if for every positive number ε and $x \in E$, there corresponds a positive integer N (depending on ε and x as well) such that

$|f_n(x) - f(x)| < \in$ whenever $n \geq N$). In this case we say that f is the pointwise limit of $\{f_n\}$ on E, and we write $\lim_{n\to\infty} f_n = f$(pointwise) for $n = 1,2,3...$

ii. $For\ n = 1,2,3 \ldots \infty$, let $S_n(x) = f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x)\ for\ x \in E$. If the sequence $\{S_n\}$ of functions (called the partial sums of $\sum_{n=1}^{\infty} f_n$) converges to f pointwise on E, we say that the series $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ for every $x \in E$, and we write it as $\sum_{n=1}^{\infty} f_n = f$ (pointwise).

iii. We say that the sequence $\{f_n\}$ converges uniformly to f on E is every positive number $\in$ there corresponds a positive integer N such that $|f_n(x) - f(x)| < \varepsilon$ when ever $n \geq N$ and for all $x \in E$. In this case we say that f is the uniform limit of $\{f_n\}$ and write it as $\lim_{n\to\infty} f_n(x) = f(x)$ $(x \in E)$ or $\lim_{n\to\infty} f_n = f$ uniformly on E.

iv. We say that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on E if the sequence $\{S_n\}$ of partial seems converges uniformly on E to f i.e., for every positive number ε there corresponds a positive integer N such that $|S_n(x) - f(x)| < \in$ whenever $n \geq N$ and for all $x \in E$.

1.3 LEMMA:

If $\{f_n\}$ converges uniformly to *f* on *E,* then $\{f_n\}$ converges pointwise to *f* on *E.*

Proof: Let $x_0 \in E$ take $\varepsilon > 0$

Claim: $f_n(x_0) \to f(x_0)$ as $n \to \infty$

Since $f_n \to f$ uniformly on E, $\exists$ a positive integer N such that $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$ for all $x \in E$

In particular, $|f_n(x_0) \to f(x_0)| < \varepsilon$ whenever $n \geq N$

This shows that $\lim_{n\to\infty} f_n(x_0) = f(x_0)$

That is, $\{f_n\}$ converges pointwise to *f* on *E.*

Remark: The converse of the above Lemma is not true.

Justification: For $n = 1,2,3 \ldots \infty$

Define $f_n: (0,1) \to \mathbb{R}$ by $f_n(x) = \frac{n}{nx+1}$ for all *x* in $(0,1)$.

Then $\forall\, x \in (0,1)$ $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{n}{nx+1} = \lim\limits_{n\to\infty} \frac{n}{n\left(x+\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \frac{1}{x+\frac{1}{n}} = \frac{1}{x} = f(x)$(say)

Now we show that this convergence is not uniform.

Consider $|f_n(x) - f(x)| = \left|\frac{n}{nx+1} - \frac{1}{x}\right| = \left|\frac{nx-(nx+1)}{x(nx+1)}\right| = \frac{1}{(nx+1)x}$

Take $\varepsilon > 0$

Claim: $\exists$ a positive integer N such that $|f_n(x) - f(x)| <\in \forall\ n \geq N$

$1 < (nx+1)x$ for all $n \geq N$ for all x

$\Leftrightarrow 1 < nx^2 + x$ for all $n \geq N$ for all x

$\Leftrightarrow 1 - x \leq nx^2$ for all $n \geq N$ for all x

$\Leftrightarrow \frac{1-x}{x^2} \leq n$ for all $n \geq N$ for all x

$\Leftrightarrow \frac{1}{x^2} - \frac{1}{x} \leq n$ for all $n \geq N$ for all x

Let $x = \frac{1}{2N}$

Therefore $\frac{1}{x^2} - \frac{1}{x} = 4N^2 - 2N = 2N(2N-1) > N$

Let N be a smallest positive integer such that N > $\frac{1-\varepsilon x}{\varepsilon x^2} + 1$

Then N is a positive integer and $|f_n(x) - f(x)| = \frac{1}{(nx+1)x} < \varepsilon\ \forall\ n \geq N\ \ \forall\, x \in E$

Thus the convergence of $\{f_n\}$ is pointwise convergence, but not uniform.

1.3.1 Example: The following is an example concerns a "double sequence":

For m=1,2,3….; n=1,2,3… , let $s_{m,n} = \frac{m}{m+n}$

Now for every fixed n, $s_{m,n} = \frac{m}{m+n} = \frac{m}{m(1+(\frac{n}{m}))} = \frac{1}{(1+\frac{n}{m})}$

So, for every fixed n, $\lim\limits_{m\to\infty} s_{m,n} = \lim\limits_{m\to\infty} \frac{1}{1+(\frac{n}{m})} = 1$

$\therefore \lim\limits_{n\to\infty} \lim\limits_{m\to\infty} s_{m,n} = \lim\limits_{n\to\infty} (1) = 1$

On the other hand for every fixed m, $\lim\limits_{n\to\infty} s_{m,n} = \lim\limits_{n\to\infty} \frac{m}{m+n} = 0$

$\therefore \lim_{m\to\infty}\lim_{n\to\infty} s_{m,n} = \lim_{m\to\infty}(0) = 0$

Hence, $\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} \neq \lim_{m\to\infty}\lim_{n\to\infty} s_{m,n}$.

1.3.2 Example: $m = 1,2,3..$ let $f_m(x) = \lim_{n\to\infty}(\cos(m!\,\pi x))^{2n}$

If $m!\,x = k$, an integer, then $f_m(x) = \lim_{n\to\infty}(\cos(kx))^{2n} = \lim_{n\to\infty}(\pm 1)^{2n} = \lim_{n\to\infty}(1) = 1$

If $m!\,x$ is not an integer, then $f_m(x) = \lim_{n\to\infty}\left(\cos(m!\,\pi x)\right)^{2n} = 0$

Now let $f(x) = \lim_{m\to\infty} f_m(x)$ $\because 0 < \cos(m!\,\pi x) < 1\ if\ m!\,x$ is not a limit

If x is irrational then $f_m(x) = 0\ \forall\ m$ and hence

$$f(x) = \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty}(0) = 0$$

Suppose x is rotational then $x = \frac{p}{q}$ where p, q are integer and $q \neq 0$

For every m, $m!\,x = m!\frac{p}{q}$ is an integer if $m \geq q$ so that

$f_m(x) = \lim_{n\to\infty}\left(\cos(m!\,\pi x)\right)^{2n} = 0 = \lim_{n\to\infty}(1) = 1$ and hence

$f(x) = \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty}(1) = 1$.

$f(x) = \lim_{m\to\infty}\lim_{n\to\infty}\left(\cos(m!\,\pi x)\right)^{2n} = \begin{cases} 0, if\ x\ is\ irrational \\ 1, if\ x\ is\ rational \end{cases}$

We know that if $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, then $f \notin \mathbb{R}$ on $[a, b]$ for any $a < b$

Thus the limit function f is discontinuous everywhere and not Riemann integrable.

1.3.3 Example: For $n = 1,2\ldots.$ let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ (x real), and let $f(x) = \lim_{n\to\infty} f_n(x)$

Then $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{\sin nx}{\sqrt{n}} = 0$

$(\because \frac{\sin nx}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}\ for\ n = 1,2\ldots and\ \frac{1}{\sqrt{n}} \to 0\ as\ n \to \infty)$

Now $f'(x) = 0\ \forall\ x \in \mathbb{R}$ $f_n'(x) = \frac{n cos nx}{\sqrt{n}} = \sqrt{n}\cos nx$

Since $\lim_{n\to\infty}\cos nx$ does not exist, we have that $f_n' \nrightarrow f'$ as $n \to \infty$

That is, $\{f_n'\}$ does not converge to $'$.

(for instance, $f_n'(0) = \frac{ncosn(0)}{\sqrt{n}} = \sqrt{n}$

$\therefore \lim_{n\to\infty} f_n'(0) = \infty, where\ f'(0) = 0$ $(\because \sqrt{n} \to +\infty$ as $n \to \infty))$

1.3.4 Example: For $n = 1,2,3\ldots.\ let\ f_n(x) = n^2x(1-x^2)^n\ (0 \le x \le 1)$ (1)

Now for $n = 1,2,3\ldots.\ f_n(0) = 0$ and $f_n(1) = 0$

$\therefore \lim_{n\to\infty} f_n(0) = 0\ and\ \lim_{n\to\infty} f_n(1) = 0$ (2)

For $0 < x < 1$, we have $f_n(x) = n^2x(1-x^2)^n = \frac{n^2x}{(1-x^2)^{-n}} = \frac{n^2x}{(\frac{1}{1-x^2})^n} = \frac{n^2x}{(1+\frac{x^2}{1-x^2})^n}$

We know that if $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ (Theorem 3.20 d)

So, $\lim_{n\to\infty} f_n(x) = x \lim_{n\to\infty} \frac{n^2}{(1+\frac{x^2}{1-x^2})^n} = 0 (\because 0 < x < 1 \Rightarrow \frac{x^2}{1-x^2} > 0)$

$\lim_{n\to\infty} f_n(x) = 0\ (0 < x < 1)$ (3)

From 2 & 3 , $\lim_{n\to\infty} f_n(x) = 0 (0 \le x \le 1)$

Consider $\int_0^1 x\,(1-x^2)^n dx = \int_{t=1}^0 t^n \left(\frac{-1}{2}\right) dt = \frac{-1}{2}\int_1^0 t^n\,dt$

$= \frac{-1}{2}\left[\frac{t^{n+1}}{n+1}\right]_1^0 = \frac{-1}{2}\left[0 - \frac{1}{n+1}\right] = \frac{1}{2(n+1)}$

$\therefore\ for\ n = 1,2,3\ldots, \int_0^1 f_n(x)\,dx = \int_0^1 n^2x(1-x^2)^n\,dx = n^2\int_0^1 x(1-x^2)^n\,dx = \frac{n^2}{2n+2}$

Now $\int_0^1 f_n(x)\,dx = \frac{n^2}{2n+2} \to \infty\ as\ n \to \infty$

If, in equation 1, we replace $n^2\ by\ n$, then $\lim_{n\to\infty} f_n(x) = 0\ , 0 \le x \le 1$

So that $\int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx = \int_0^1 (0)\,dx = 0$

But $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{2n+2} = \lim_{n\to\infty} \frac{n}{n(2+\frac{2}{n})} = \lim_{n\to\infty} \frac{1}{2+\frac{2}{n}} = \frac{1}{2}$

$$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both are finite.

Note: Give an example to show that a convergent series of continuous functions may have a discontinuous term.

1.3.5 Example: Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for $n = 1,2,3\ldots.$ for $x \in \mathbb{R}$

For $n = 1,2,3\ldots. x \in \mathbb{R}$ write $S_n(x) = \sum_{k=0}^{n} f_k(x)$ for $x \in \mathbb{R}$

We know that for $x \in \mathbb{R}$ with $x \neq 1$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

Observe that for every $0 \neq x \in \mathbb{R}$, $0 < \frac{1}{1+x^2} < 1$

$$\text{For } 0 \neq x \in \mathbb{R},\ S_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}$$

$$= x^2 \sum_{k=0}^{n} \frac{1}{(1+x^2)^k}$$

$$= x^2 \left\{\frac{1-(\frac{1}{1+x^2})^{n+1}}{1-\frac{1}{1+x^2}}\right\} \qquad (\because \forall\, 0 \neq x \in \mathbb{R}, \frac{1}{1+x^2} \neq 1)$$

$$= x^2 \left\{\frac{\left(\frac{(1+x^2)^{n+1}-1}{(1+x^2)^{n+1}}\right)}{\left(\frac{(1+x^2)-1}{(1+x^2)}\right)}\right\}$$

$$= x^2 \left\{\frac{(1+x^2)^{n+1}-1}{(1+x^2)^{n+1}}\ X\ \frac{1+x^2}{x^2}\right\}$$

$$= (1+x^2)\left\{1 - \frac{1}{(1+x^2)^{n+1}}\right\}$$

So, $\lim_{n\to\infty} \left(\frac{1}{1+x^2}\right)^n = 0$

$\therefore \lim_{n\to\infty} S_n(x) = 1 + x^2\ if\ x \neq 0 \quad (f|x| < 1, then\ \lim_{n\to\infty} x^n = 0)$

Also clearly $\lim_{n\to\infty} S_n(0) = 0$

For $x \in \mathbb{R}$, define $f(x) = \{^{0 \quad if\ x=0}_{1+x^2 \quad if\ x\neq 0},$

Then $\sum_{n=0}^{\infty} f_n(x) = \lim_{n\to\infty} S_n(x) = f(x)\ (x \in \mathbb{R})$

The series converges point wise on $\mathbb{R}$ to f.

Now we prove that the convergence is not uniform on $\mathbb{R}$.

If possible suppose that the convergence is uniform on $\mathbb{R}$.

Then $\{S_n\}$ converges uniformly on $\mathbb{R}$ to f.

So corresponding to $\varepsilon = \frac{1}{2}$, there exists a positive integer such that

$$|S_n(x) - f(x)| < \frac{1}{2} \qquad (1)$$

Whenever $n \geq N$ for all $x \in \mathbb{R}$ in particular (1) is true for $n = N$ and $x \neq 0$

So we have $|S_n(x) - f(x)| < \frac{1}{2} \Rightarrow \left|(1+x^2)\left\{1 - \frac{1}{(1+x^2)^{n+1}}\right\} - (1+x^2)\right| < \frac{1}{2}\ \forall x \neq 0$

$\Rightarrow \frac{1}{(1+x^2)^N} < \frac{1}{2}$ for all $x \neq 0 \Rightarrow (1+x^2)^N > 2\ \ \forall x \neq 0$

$\Rightarrow x^2 > 2^{\frac{1}{N}} - 1\ \forall x \neq 0$

$\Rightarrow |x| > \left(2^{\frac{1}{N}} - 1\right)^{\frac{1}{2}}\ \forall x \neq 0$

$\therefore \mathbb{R} - \{0\} \subseteq \left\{x \mid |x| > \left(2^{\frac{1}{N}} - 1\right)^{\frac{1}{2}}\right\},$

Which is not possible.

Hence the convergence is not uniform on $\mathbb{R}$.

1.4 UNIFORM CONVERGENCE:

In this section, we introduce a stronger notion of convergence of functions than point wise convergence, called uniform convergence. The difference between point wise convergence and uniform convergence is analogous to the difference between continuity and uniform continuity.

1.4.1 Definition: we say that a sequence of functions $\{f_n\}$, $n = 1,2,3, \ldots$, converges uniformly on E to a function f if for every $\varepsilon > 0$ there is an ingeter N such that $n \geq N$ implies

$|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in E$ (1)

It is clear that every uniformly convergent sequence is point wise convergent. Quite explicitly, the difference between the two concepts in this: if $\{f_n\}$ converges point wise on E, then there exists a function f such that, for every $\varepsilon > 0$, and for every $x \in E$, there is an integer N, depending on ε and on x, such that (1) holds if $n \geq N$; if $\{f_n\}$ converges uniformly on E, it is possible, for each $\varepsilon > 0$, to find one interger N which will do for all $x \in E$.

We say that the series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$\sum_{i=1}^{n} f_i(x) = s_n(x)$

Converges uniformly on E.

1.4.2 Theorem: Cauchy criterion for uniform convergence of sequence of functions.

Statement: Let $\{f_n\}$ be a sequence of functions defined on *E*. Then the sequence $\{f_n\}$ convergence uniformly on *E* if and only if for every $\varepsilon > 0$ there exist an integer N such that $m \geq N, n \geq N, x \in E$ implies $|f_n(x) - f_m(x)| \leq \varepsilon$

Proof: Suppose $\{f_n\}$ converges uniformly on *E*

Let f be its limit function.

Then for every $\varepsilon > 0$ there corresponds a positive integer N such that implies $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$

So that $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ if $m \geq N,\ n \geq N, x \in E$

Conversely, suppose that the Cauchy condition holds.

i.e., for every real number $\varepsilon > 0$, there corresponds a positive integer N such that $|f_n(x) - f_m(x)| \leq \varepsilon\ m \geq N, n \geq N$ for all $x \in E$

then for every $x \in E$, $\{f_n(x)\}$ is a sequence of numbers that satisfies cauchy's general principle for convergence.

So, there exists a number $f(x)$ depending on x to which $f_n(x)$ converges.

Clearly, $f(x)$is uniquely fixed with x ($\because$ limit of a sequence is unique)

Now $x \to f(x)$ defines a function on E and $f(x) = \lim_{n\to\infty} f_n(x)$

To show that the convergence is uniform.

Take $\varepsilon > 0$

By our Supposition there exists a positive integer N such that $|f_n(x) - f_m(x)| \leq \varepsilon$ whenever $m \geq N, n \geq N$ for all $x \in E$

Fix m, and let n → ∞

Then we get $\left|\lim_{n\to\infty} f_n(x) - f_m(x)\right| \leq \varepsilon$ for every $m \geq N$ for every $x \in E$

So, from (1) $|f(x) - f_m(x)| \leq \varepsilon$ for every $m \geq N$ for every $x \in E$

Hence, $\{f_m\}$ converges uniformly on E to f (or) $\{f_n\}$Converges uniformly on E to f.

1.4.3 Theorem: Suppose $\lim_{n\to\infty} f_n(x) = f(x)\ x \in E$ Put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$

Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$

Proof: Suppose $\lim_{n\to\infty} M_n = 0$ (1)

Take $\varepsilon > 0$

Then from (1), there exists a positive integer N depending on E such that

$0 \leq M_n \leq \varepsilon$ whenever $n \geq N$

This implies that for every $x \in E$ and $n \geq N$, $|f_n(x) - f(x)| \leq M_n < \varepsilon$

This shows that $\{f_n\}$ converges uniformly on E to f.

Conversely, suppose that sequence $\{f_n\}$ converges uniformly on E, to f

Choose $\varepsilon > 0$

Then by our supposition, ∃ a positive integer N, depending on ∈ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ whenever $\geq N\ \forall\ x \in E$.

This implies that $0 \leq M_n = \left\{ \sup_{x \in E} |f_n(x) - f(x)| \right\} \leq \frac{\varepsilon}{2} < \epsilon \ \forall\, n \geq N$

Hence, $\lim_{n \to \infty} M_n = 0$

1.4.4 Theorem: (Weierstross M-Test for uniform convergence)

Statement: Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$|f_n(x)| \leq M_n \ (x \in E, n = 1,2,3 \ldots)$ Then $\sum f_n$ convergence uniformly on E if $\sum M_n$ converges.

Proof: Given that $|f_n(x)| \leq M_n \ (x \in E, n = 1,2,3 \ldots)$ (1)

For $x \in E$, let $S_n(x) = f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x) \ \ \forall n \geq 1$

And let $U_n = M_1 + M_2 + \ldots + M_n \ \forall n \geq 1$

Suppose $\sum_{n=1}^{\infty} M_n$ converges. Then the partial terms sequence $\{U_n\}$ is convergent.

We know that every convergent sequence in a metric space is a Cauchy sequence so $\{U_n\}$ is a Cauchy sequence (2)

Take $\varepsilon > 0$

then from (2) there exists a positive integer N such that $|U_m - U_n| < \varepsilon$ whenever $m > n \geq N$

Now for $m > n \geq N$ and $x \in E$,

$$|S_m(x) - S_n(x)| = |(f_1(x) + f_2(x) + \ldots + f_m(x)) - (f_1(x) + f_2(x) + \ldots + f_n(x))|$$

$$= |(f_{n+1}(x) + f_{n+2}(x) + \ldots + f_m(x))|$$

$$= |f_{n+1}(x)| + |f_{n+2}(x)| + \ldots + |f_m(x)|$$

$$= M_{n+1} + M_{n+2} + \ldots . + M_m$$

$$= |U_m - U_n|$$

$$< \varepsilon$$

$\therefore$ By theorem (1), $\{S_n\}$ converges uniformly on E.

Hence $\sum_{n=1}^{\infty} f_n$ converges uniformly on E, to some function defined on E.

1.5 SUMMARY:

In the present Lesson we confine our attention to point wise convergence, related theorems and examples as well as uniform convergence, related theorems and examples.

1.6 TECHNICAL TERMS:

1. Uniform convergence
2. Point wise convergence
3. Weierstrass M-Test

1.7 SELF-ASSESSMENT QUESTIONS:

1. Prove that every uniformly convergent sequence of bounded is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converges uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.
3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converges uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converges on E).

1.8 SUGGESTED READINGS:

1. Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985.

- Dr. K. Gangadhar

LESSON- 2
SEQUENCES AND SERIES OF FUNCTIONS: UNIFORM CONVERGENCE AND CONTINUITY

OBJECTIVES:

After studying the lesson you should able to understand the concept of uniform convergence and continuity. Definition of Normed Linear Space.

STRUCTURE:

2.1 Introduction

2.2 Uniform convergence and continuity

2.3 Summary

2.4 Technical terms

2.5 Self- Assessment Questions

2.6 Suggested readings

2.1 INTRODUCTION:

In this lesson, we define and study the uniform convergence and continuity, and also integration of sequences and series of functions. There are many different ways to define the uniform convergence of a sequence of functions, and different definitions lead to in equivalent types of convergence.

2.2 UNIFORM CONVERGENCE AND CONTINUITY:

2.2.1 Theorem: Suppose $f_n \to f$ uniformly on a set E in a metric space (x, d). Let x be a limit point of E, and suppose that $\lim_{t\to x} f_n(t) = A_n \ (n = 1,2,\dots)$. Then $\{A_n\}$ converges, and $\lim_{t\to x} f_n(t) = \lim_{n\to\infty} A_n$ In other words, the conclusion is that

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Proof: Suppose $f_n \to f$ uniformly on E and $\lim_{t\to x} f_n(t) = A_n (n = 1,2,\dots)$

Claim: $\{A_n\}$ is a Cauchy's sequence

Let x be a limit point of E

Let $\varepsilon > 0$ be given

Then by Cauchy's general principle for uniform convergence, there exists a positive integer N_1 such that $|f_n(t) - f_m(t)| \leq \frac{\varepsilon}{2}$ for $n \geq N_1, m \geq N_1$ $and\ t \in E$ (1)

Letting $t \to x$ in (1), we obtain $\left|\lim_{t\to x} f_n(t) - \lim_{t\to x} f_m(t)\right| \leq \frac{\varepsilon}{2}$ for $n \geq N_1, m \geq N_1$

So, Since $\lim_{t\to x} f_k(t) = A_k (k = 1,2,\ldots)$, it follows that

for $n \geq N_1, m \geq N_1$, $|A_n - A_m| \leq \frac{\varepsilon}{2} < \epsilon$

Hence, $\{A_n\}$ is a Cauchy sequence and hence converges, say to A i.e., $\lim_{n\to\infty} A_n = A$

In $\mathbb{R}^k$, every cauchy sequence is convergent

Claim: $\exists$ a positive integer N and $\exists$ a $\delta > 0$ such that

i) $|f_N(t) - f(t)| \leq \frac{\varepsilon}{3}\ \forall\ t \in E$

ii) $|A_n - A| \leq \frac{\varepsilon}{3}$ and

iii) $|f_N(t) - A_N| \leq \frac{\varepsilon}{3}\ if\ t \in E\ 0 < d(t,x) < \delta$

We first choose positive integers $N_2 and\ N_3$ such that

$|f_n(t) - f(t)| \leq \frac{\varepsilon}{3}$ for $n \geq N_2$ and $\forall\ t \in E$ and $|A_n - A| \leq \frac{\varepsilon}{3}$ for $n \geq N_3$

($\because\ f_n \to f\ uniformly\ on\ E\ and \lim_{n\to\infty} A_n = A$)

Put $N\ = max\{N_2, N_3\}$

Then we have $|f_N(t) - f(t)| \leq \frac{\varepsilon}{3}\ \forall\ t \in E$ and $|A_N - A| \leq \frac{\varepsilon}{3}$

Since $\lim_{t\to x} f_N(t) = A_N, \exists\ a\ \delta > 0$ such that $0 < d(t,x) < \delta\ t \in E$

implies $|f_N(t) - A_N| \leq \frac{\varepsilon}{3}$

Claim: $\lim_{t\to x} f(t) = A$

Choose a $\delta > 0$ and a positive integer N satisfying (i),(ii) and (iii)

If $0 < d(t,x) < \delta\ t \in E$

$\therefore |f(t) - A| \leq |f(t) - f_N(t)| + |f_N(t) - A_N| + |A_N - A| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

When ever $0 < d(t,x) < \delta \ for\ t \in E.$

Hence, $\lim_{t \to x} f(t) = A$ that is $\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$

2.2.2 Theorem: Let (x,d) be a metric space and $E \subseteq X$. If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.

Proof: Let $x \in E$ and $\varepsilon > 0$

Since $\{f_n\}$ converges uniformly on E to f, there exists a positive integer N such that $|f_n(y) - f(y)| \leq \frac{\varepsilon}{3}$ for $n \geq N$ and $\forall\ y \in E$

In particular, $|f_N(y) - f(y)| \leq \frac{\varepsilon}{3}\ \forall\ y \in E$ (1)

Since f_N is continuous at , there exists $\delta > 0$ such that

$0 < d(x,y) < \delta \Rightarrow |f_N(y) - f_N(y)| \leq \frac{\varepsilon}{3} \forall\ y \in E$ (2)

$$\therefore |f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Whenever $0 < d(x,y) < \delta\ for\ x \in E$ (by (1)&(2))

Hence, f is continuous at x

This is true for every $x \in E$

$\therefore f$ is continuous on E.

Remark: The converse of the above theorem is not true.

That is, a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform.

Justification:

Example: Define $f_n: [0,1] \to \mathbb{R}$ by $f_n(x) = n^2x(1-x^2)^n, 0 \leq x \leq 1$

Then $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} n^2x(1-x^2)^n = \lim_{n \to \infty} \frac{n^2x}{\left(\frac{1}{(1-x^2)^n}\right)}$

$$= x \lim_{n \to \infty} \frac{n^2}{\left(1+\frac{x^2}{1-x^2}\right)^n} = x(0) = 0;\ 0 < x < 1$$

$\because 0 < x < 1 \Rightarrow \frac{x^2}{1-x^2} > 0 \lim_{n\to\infty} \frac{n^2}{(1+p)^n} = 0 \, for \;\; p > 0, \alpha \in \mathbb{R}$

Now $f_n(0) = 0$ & $f_n(1) = 0$

$\therefore \lim_{n\to\infty} f_n(x) = 0, 0 \le x \le 1$

For $0 \le x \le 1$, define $f(x) = 0$

Then f is continuous on [0,1] and $\lim_{n\to\infty} f_n(x) = f(x)$

Also, each f_n is continuous on [0,1] for $n = 1,2,\ldots$

Put $M_n = \sup_{0\le x\le 1} |f_n(x) - f(x)| = \sup_{0\le x\le 1} f_n(x)$

Now $f_n'(x) = n^2(1-x^2)^n + n^2x(-2x).\,n(1-x^2)^{n-1}$

$f_n'(x) = 0 \Rightarrow (1-x^2)^{n-1}n^2[(1-x^2) - 2x^2n] = 0$

$\Rightarrow (1-x^2)^{n-1} = 0 \quad , \quad 1 - x^2 - 2x^2n = 0$

$\Rightarrow 1 - x^2 = 0 \quad , \quad 1 - (1+2n)x^2 = 0$

$\Rightarrow x = \pm 1 \quad , \quad x = \pm\frac{1}{\sqrt{1+2n}}$

As $x \ge 0$, we consider $x_1 = +1$, $x_2 = \frac{1}{\sqrt{1+2n}}$

Now $f_n''(x) = n^2(n-1)(1-x^2)^{n-2}(-2x)[(1-x^2) - 2x^2n] + (1-x^2)^{n-1}n^2[-2x - 4nx]$

For $x_1 = 1, f_n''(x_1) = f_n''(1) = 0$

So, we cannot decide anything at $x_1 = 1$

For $x_2 = \frac{1}{\sqrt{1+2n}}$,

$f_n''(x_2) = f_n''(\frac{1}{\sqrt{1+2n}}) = n^2(n-1)\left(1 - \frac{1}{2n+1}\right)^{n-2}(-2n)\left[1 - \frac{1}{2n+1} - \frac{2n}{2n+1}\right] + \left(1 - \frac{1}{2n+1}\right)^{n-1}(n^2)\left[-2\frac{1}{\sqrt{2n+1}} - \frac{4n}{\sqrt{2n+1}}\right]$

$= -2\left(\frac{2n}{2n+1}\right)^{n-1} n^2 \left(\frac{1+2n}{\sqrt{1+2n}}\right) < 0$

$\therefore$ f_n has maximum at $\frac{1}{\sqrt{2n+1}}$ and the maximum value of f_n is

$f_n\left(\frac{1}{\sqrt{1+2n}}\right) = n^2 \frac{1}{\sqrt{1+2n}}\left(1-\frac{1}{2n+1}\right)^n = \frac{n^2}{\sqrt{1+2n}}\left(\frac{2n}{2n+1}\right)^n$

For $n = 1,2,3\ldots$

$$M_n = \sup_{0\le x\le 1}|f_n(x)-f(x)| = \left|\frac{n^2}{\sqrt{1+2n}}\left(\frac{2n}{2n+1}\right)^n - 0\right|$$

$$= \frac{2^n.n^{n+2}}{(2n+1)^{n+\frac{1}{2}}} = \frac{2^n.n^{n+2}}{n^{n+\frac{1}{2}}\left(2+\frac{1}{n}\right)^{n+\frac{1}{2}}} = \frac{2^n.n^{3/2}}{\left(2+\frac{1}{n}\right)^{1/2}\left(2+\frac{1}{n}\right)^n}$$

$$= \frac{2^n.n^{3/2}}{2^n\left(2+\frac{1}{n}\right)^{1/2}\left(1+\frac{1}{2n}\right)^n} = \frac{n^{3/2}}{\sqrt{2+\frac{1}{n}}\left(1+\frac{1}{2n}\right)^n}$$

So, $M_n \to \infty$ as $n \to \infty$

That is $M_n \nrightarrow 0$ as $n \to \infty$

Thus $f_n \to f$ is not uniform on [0,1]. (by theorem 1.11)

2.2.3 Theorem: Suppose K is a compact subset of a metric space (x,d), and

a) $\{f_n\}$ is a sequence of continuous functions on K,
b) $\{f_n\}$ converges pointwise to a continuous function of on K
c) $f_n(x) \ge f_{n+1}(x)\ \forall\ x \in K, n = 1,2,3\ldots$
   Then $f_n \to f$ uniformly on K.

Proof: For $n \ge 1$ write $g_n(x) = f_n(x) - f(x)\forall x \in K$

Since each f_n and f are continuous, we have that each g_n is a continuous function on K.

Since, $\lim_{n\to\infty} f_n(x) = f(x)\forall x \in K,\ \ \lim_{n\to\infty} g_n(x) = 0\ \forall x \in K$

Also, $g_n(x) \ge g_{n+1}(x)\forall x \in K, n = 1,2,3,\ldots$ as $f_n(x) \ge f_{n+1}(x)\forall x \in K$

We have to prove that the sequence $\{g_n\}$ converges uniformly on K to 0.

Let $\in > 0$ and $x \in K$.

Since, $\lim_{n\to\infty} g_n(x) = 0$, $\exists$ a positive integer $N(\varepsilon,x)$ such that

$0 \le g_n(x) < \frac{\varepsilon}{2}$ for $n \ge N(\varepsilon,x)$

We denote this $N(\in,x)$ by $N(x)$.

In particular, $0 \leq g_{N(\varepsilon,x)} = g_{N(x)} < \frac{\varepsilon}{2}$

Since, $g_{N(x)}$ is continuous at x, $\exists\, a$ real number $\delta > 0$ (depending on x) such that $\left|g_{N(x)}(y) - g_{N(x)}(x)\right| < \frac{\varepsilon}{2}$ for $y \in K$ such that $d(x,y) < \delta$ (1)

Put $J(x) = \{y \mid y \in K \; and \; d(x,y) < \delta\}$

Then clearly $J(x)$ is an open set in K ($\because J(x)$ is neighbourhood)

Now the family $\{J(x) \mid x \in K\}$ is an open cover of the compact space K, so that there exist finitely many x in K, say $x_1, x_2, \ldots . x_r$ such that $K = \overset{r}{\underset{i=1}{U}} J(x_i)$

Write $N(\in) = max\{N(x_1), \ldots N(x_r)\}$

It is clear that $N(\varepsilon)$ depends on ε only

Let $y \in K$

Then $y \in J(x_i)$ for some $1 \leq i \leq r$

So, $\left|g_{N(xi)}(y) - g_{N(xi)}(xi)\right| < \frac{\varepsilon}{2}$ (by (1))

$\Rightarrow g_{N(xi)}(xi) - \frac{\varepsilon}{2} < g_{N(xi)}(y) < g_{N(xi)}(xi) + \frac{\varepsilon}{2}$

Since $g_n(y) \leq g_{N(xi)}(y)$ for $n \geq N(xi)$ and $g_{N(xi)}(y) < g_{N(xi)}(xi) + \frac{\varepsilon}{2} < \varepsilon$,

We have that $g_n(y) < \varepsilon$ for $n \geq N(xi)$

If $n \geq N(\varepsilon)$, then $n \geq N(xi)$ for $i = 1,2,3, \ldots . r$

$\therefore$ for all $y \in K$, $0 \leq g_n(y) < \varepsilon$ whenever $n \geq N(\varepsilon)$.

Hence $\{g_n\}$ converges uniformly on K to 0

That is, $f_n \to f$ uniformly on K.

Example: For $= 1,2,3\ldots$, define $f_n(x) = \frac{1}{nx+1}, 0 < x < 1$

Then each f_n is continuous on $(0,1)$

We have $n < n + 1 \Rightarrow nx < (n + 1)x \Rightarrow nx + 1 < (n + 1)x + 1$

$\Rightarrow \frac{1}{nx+1} > \frac{1}{(n+1)x+1} \Rightarrow f_n(x) > f_{n+1}(x) \; \forall \, x \; \& \; \forall \, n$

Also, $f_n(x) \to 0$ as $n \to \infty$ (1)

Let $\varepsilon > 0$ be given

Then by (1), $\exists$ a positive integer N such that $|f_n(x)| < \varepsilon \; \forall \; n \geq N$

$\Rightarrow \frac{1}{nx+1} < \varepsilon \Rightarrow nx + 1 > \frac{1}{\varepsilon} \Rightarrow nx > \frac{1}{\varepsilon} - 1 \Rightarrow n > \left(\frac{1}{\varepsilon} - 1\right)\frac{1}{x}$

Choose $N = \left(\frac{1}{\varepsilon} - 1\right)\frac{1}{x} + 1$. Then $\forall \; n \geq N$, we have $|f_n(x)| = \frac{1}{nx+1} < \varepsilon$

Observe that N depends on both ε & x.

$\therefore$ the convergence of $\{f_n\}$ to 0 is not uniform

Hence, compactness of interval is really needed.

2.2.4 Definition: Normed Linear Space: Let X be a vector space over scalar field K. A map $\|.\|: X \to \mathbb{R}$

Is called as norm (on X) if it satisfies the following conditions

i) $\| x \| \geq 0$, and $\| x \| = 0 \Leftrightarrow x = 0$
ii) $\| \alpha x \| = |\alpha| \, \| x \|$ for all $\alpha \in K$ & $x \in X$
iii) $\| x + y \| \leq \| x \| + \| y \| \; \forall \; x, y \in X$

Here $(X, \|.\|)$ is called a normed linear space.

2.2.5 Lemma: Every normed linear space $(X, \|.\|)$ is a metric space with respect to the metric 'd' defined by $d(x, y) = \| x - y \| \; \forall \; x, y \in X$.

Proof: Let X be a normed linear space.

Let $x, y, z \in X$

i) $d(x, y) = \| x - y \| \geq 0$ and $d(x, y) = 0 \Leftrightarrow \| x - y \| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$
ii) $d(x, y) = \| x - y \| = \| -(y - x) \| = |-1| \, \| y - x \| = \| y - x \| = d(y, x)$
iii) $d(x, y) = \| x - y \| = \| x - z + z - y \| \leq \| x - z \| + \| z - y \| = d(x, z) + d(z, y)$

$\therefore (X, d)$ is a metric space.

2.2.6 Definition: Let X be a normed linear space. Let $\{x_n\}$ be a sequence in X.

i) We say that the sequence $\{x_n\}$ converges to a point x in X. if given $\varepsilon > 0$, $\exists$ a positive integer N such that $\| x_n - x \| < \varepsilon \; \forall \; n \geq N$

ii) We say that the sequence $\{x_n\}$ is a Cauchy sequence if given $\varepsilon > 0$, $\exists$ a positive integer N such that $\| x_n - x_m \| < \varepsilon \; \forall \, n \geq N \; \& \; \forall \, m \geq N$.

2.2.7 Definition: A normed linear space X is said to be complete if every Cauchy sequence in X is convergent to an element of X.

2.2.8 Notation: If X is a metric space then the set of all complex- valued, continuous, bounded functions with domain X is denoted by $\mathbb{C}(X)$.

2.2.9 Result: $\mathbb{C}(X)$ is a normed linear space.

Proof: Define '+' & '.' on $\mathbb{C}(X)$ as follows

For any $f, g \in \mathbb{C}(X)$, $(f+g)(x) = f(x) + g(x)$ and

$(\alpha f)(x) = \alpha f(x) \; \forall \alpha \in K \; \& \; \forall \, x \in X$

Now $\mathbb{C}(X)$ is a vector space under the above binary operation '+' and scalar multiplication '.'

For any $f \in \mathbb{C}(X)$, define $\| f \| = \sup_{x \in X} |f(x)|$

Let $f \in \mathbb{C}(X)$

Then f is bounded on X, so that $\sup_{x \in X} |f(x)|$ exists and hence $\| f \|$ exists.

i) $\| f \| = 0 \Leftrightarrow \sup_{x \in X} |f(x)| = 0 \Leftrightarrow |f(x)| = 0 \; \forall \, x \in X$

$$\Leftrightarrow f(x) = 0$$

$$\Leftrightarrow f = 0$$

ii) Since $|f(x)| \geq 0 \forall \, x \in X$, $\| f \| = \sup_{x \in X} |f(x)| \geq 0$

iii) Let $\alpha \in K$

Now $\| \alpha f \| = \sup_{x \in X} |(\alpha f)(x)| = \sup_{x \in X} |\alpha f(x)| = |\alpha| \sup_{x \in X} |f(x)| = |\alpha| \, \| f \|$

iv) Let $f, g \in \mathbb{C}(X)$

For $x \in X$, $|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)|$

$$\leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)|$$

$$= \| f \| + \| g \|$$

$\therefore \sup_{x \in X} |(f+g)(x)| \leq \| f \| + \| g \|$

$\Rightarrow \| f + g \| \leq \| f \| + \| g \|$

$\therefore (\mathbb{C}(x), \|.\|)$ is a normed linear space. $

Remark: For any $f, g \in \mathbb{C}(X)$, define $d(f,g) = \| f - g \|$ then 'd' is a metric on $\mathbb{C}(X)$ and hence $(\mathbb{C}(X), d)$ is a metric space.

2.2.10 Lemma: A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathbb{C}(X)$, X is a metric space if and only if $f_n \to f$ uniformly on X.

Proof: $f_n \to f$ with respect to the metric of $\mathbb{C}(X)$

$\Leftrightarrow$ for any $\varepsilon > 0, \exists$ a positive integer $N(\varepsilon)$ such that

$$\| f_n - f \| < \varepsilon \ \forall \ n \geq N$$

$\Leftrightarrow$ for any $\varepsilon > 0, \exists$ a positive integer $N(\varepsilon)$ such that

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon \ \forall \ n \geq N$$

$\Leftrightarrow$ for any $\varepsilon > 0, \exists$ a positive integer $N(\varepsilon)$ such that

$$|f_n(x) - f(x)| < \varepsilon \ \forall \ n \geq N \ \& \ \forall x \in X$$

$\Leftrightarrow f_n \to f$ uniformly on X.

2.2.11 Theorem: The metric space $\mathbb{C}(X)$ of all complex-valued bounded continuous functions on a metric space X is complete with respect to the uniform metric defined by $d(f,g) = \| f - g \|$

For all $f, g \in \mathbb{C}(X)$.

Proof: We know that $(\mathbb{C}(X), d)$ is a metric space.

Now we prove that $(\mathbb{C}(X), d)$ is a complete metric space.

Let $\{f_n\}$ be a Cauchy sequence in $\mathbb{C}(X)$ (1)

Let $\varepsilon > 0$ be given

Then by (1) there exists a positive integer N such that $\| f_n - f_m \| < \epsilon \ \forall \ n, m \geq N$

$\Rightarrow \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon \ \ \forall \ n, m \geq N$

$\Rightarrow |f_n(x) - f_m(x)| < \varepsilon \forall \ n, m \geq N \ and \ \forall x \in X$ (2)

This shows that for every $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence in the complete metric space $\mathbb{C}$ and hence converge to some number $f(x)$ (say)

That is $\lim_{n\to\infty} f_n(x) = f(x) \forall x \in X$

Now we show that this convergence is uniform.

Fixing n and letting $m \to \infty$ in (2)

$\left|f_n(x) - \lim_{m\to\infty} f_m(x)\right| < \varepsilon \ \forall x \in X$

$|f_n(x) - f(x)| < \varepsilon \ \forall x \in X$

$\sup_{x\in X} |f_n(x) - f(x)| < \varepsilon$

$\therefore \| f_n - f \| < \varepsilon \ \forall n, \geq N$

Hence, $\{f_n\}$ converges to f uniformly on X.

To show that $f \in \mathbb{C}(X)$.

Since $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ uniformly on X, by theorem (5), f is continuous.

Since each f_n , $n \geq 1$ is bounded, and $\{f_n\}$ converges uniformly on X, we have that $\{f_n\}$ is uniformly bounded.

So, $\exists\ M > 0$ such that $|f_n(x)| \leq M \ \forall x \in X \ and \ \forall n \geq 1$.

Since $f_n \to f$ uniformly on X, ∃ a positive integer N such that

$|f_n(x) - f(x)| < 1 \ \forall x \in X \ and \ \forall n \geq N$

In particular, $|f_N(x) - f(x)| < 1 \ \forall x \in X$

Now for any $x \in X, |f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M$

This shows that f is bounded on X and hence $f \in \mathbb{C}(X)$.

Hence, $\{f_n\}$ converges to f in $\mathbb{C}(X)$.

∴ every Cauchy sequence in $\mathbb{C}(X)$ is convergent.

Thus, $(\mathbb{C}(X), d)$ is a complete metric space.

2.3 SUMMARY:

In this Lesson we are given detailed explanation about uniform convergence and continuity of function through definitions and theorems and also, the detailed explanation about uniform convergence and integration of function through definitions and theorems.

2.4 TECHNICAL TERMS:

- Continuous function
- Metric space
- Normed linear space

2.5 SELF-ASSESSMENT QUESTIONS:

1. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ for what values of x does tha series converges absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?.
2. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \le x \le \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right), \end{cases}$$

show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

3. Prove that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

2.6 SUGGESTED READINGS:

1. Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985.

- **Dr. K. Gangadhar**

LESSON - 3

SEQUENCES AND SERIES OF FUNCTIONS: UNIFORM CONVERGENCE AND INTEGRATION

OBJECTIVES:

After studying the lesson you should able to understand the concept of uniform convergence and continuity and also uniform convergence and integration of functions.

STRUCTURE:

3.1 INTRODUCTION:

In this lesson, we define and study the uniform convergence and integration of sequences and series of functions. There are many different ways to define the uniform convergence of a sequence of functions, and different definitions lead to in equivalent types of convergence. We consider here two basic types of uniform convergences: Continuity and Integration.

3.2. DEFINITIONS:

3.2.1 Definition (Uniform Convergence):

We say that a sequence of functions $\{f_n\}, n = 1, 2, 3, \ldots$, converges uniformly on E to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon \quad (1)$$

For all $x \epsilon E$.

It is clear that every uniformly convergent sequence is point wise convergent.

If $\{f_n\}$ convergence point wise on E, then there exists a function f such that, for every $\epsilon > 0$, and for every $x\epsilon E$, there is an integer N, depending on ϵ and on x, such that (1) holds if $n \geq N$; if $\{f_n\}$ convergence uniformly on E, it is possible, for each $\epsilon > 0$, to find one integer N which will do for all $x\epsilon E$.

We say that the series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$\sum_{i=1}^{n} f_i(x) = s_n(x)$

Converges uniformly on E.

3.2.2 Definition: Let E be a set, $\{f_n\}, n = 1,2,3 \ldots \infty$ sequence of functions defined on E and let f be a function defined on E.

i. We say that the sequence $\{f_n\}$ converges to f pointwise or converges pointwise to f on E if for every $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$ (if for every positive number ε and $x \in E$, there corresponds a positive integer N (depending on ε and x as well) such that $|f_n - f(x)| < \in$ whenever $n \geq N$). In this case we say that f is the pointwise limit of $\{f_n\}$ on E, and we write $\lim_{n\to\infty} f_n = f$(pointwise) for $n = 1,2,3 \ldots$
ii. $For\ n = 1,2,3 \ldots \infty$, let $S_n = f_1(x) + f_2(x) + f_3(x) + \ldots + f_n(x)\ for\ x \in E$. If the sequence $\{S_n\}$ of functions (called the partial sums of $\sum_{n=1}^{\infty} f_n$) converges to f pointwise on E, we say that the series $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ for every $x \in E$, and we write it as $\sum_{n=1}^{\infty} f_n = f$ (pointwise).
iii. We say that the sequence $\{f_n\}$ converges uniformly to f on E for every positive number $\in$ there corresponds a positive integer N such that $|f_n(x) - f(x)| < \varepsilon$ when ever $n \geq N$ and for all $x \in E$. In this case we say that f is the uniform limit of $\{f_n\}$ and write it as $\lim_{n\to\infty} f_n(x) = f(x)$ $(x \in E)$ or $\lim_{n\to\infty} f_n = f$ uniformly on E.
iv. We say that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on E if the sequence $\{S_n\}$ of partial seems converges uniformly on E to f i.e., for every positive number ε there corresponds a positive integer N such that $|S_n(x) - f(x)| < \in$ whenever $n \geq N$ and for all $x \in E$.

3.3 UNIFORM CONVERGENCE AND INTEGRATION:

3.2.1 Theorem: State and prove uniform convergence and integration theorem.

Statement: Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathbb{R}(\alpha)$ on $[a,b]$, for $n = 1,2,3,\ldots$, and suppose $f_n \to f$ uniformly on $[a,b]$. Then $f \in \mathbb{R}(\alpha)$ on $[a,b]$, and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$

It is sufficient to prove the theorem for real $f_{n,} n \geq 1$

Claim: f is bounded

Take $\varepsilon = 1 > 0$

Since $f_n \to f$ uniformly on $[a,b]$, there exists a positive integer N

such that $|f_n(x) - f(x)| < \varepsilon \forall\, x \in [a,b]$

In particular, $|f_N(x) - f(x)| < 1 \;\forall\, x \in [a,b]$

$\Rightarrow |f(x)| - |f_N(x)| \leq |f(x) - f_N(x)| < 1 \;\forall\, x \in [a,b]$

$\Rightarrow |f(x)| < 1 + |f_N(x)| \leq 1 + M \;\forall\, x \in [a,b]$

Where M is an upper bound of $|f_N|$ on $[a,b]$

$\therefore f$ is bounded.

Let g and h be two bounded real functions and $g(x) \leq h(x) \forall\, x \in [a,b]$

For any partition $p = \{a = x_0 < x_1 <\ldots.< x_{n-1} < x_n = b\}\; of\; [a,b]$

With $m_i = g.l.b\{g(x) \mid x_{i-1} \leq x \leq x_i\}$

$M_i = l.u.b\{g(x) \mid x_{i-1} \leq x \leq x_i\}$

$m_{\mathrm{i}}' = g.l.b\{h(x) \mid x_{i-1} \leq x \leq x_i\}$

$M_i' = l.u.b\{h(x) \mid x_{i-1} \leq x \leq x_i\}\; for\; i = 1,2,3,\ldots.n$

We have $m_i \leq m_{\mathrm{i}}'$ and $M_i \leq M_i'$ $(\because g(x) \leq h(x) \forall x \in [a,b])$

$\therefore\; L(p,g,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \leq \sum_{i=1}^{n} m_{\mathrm{i}}' \Delta\alpha_i = L(p,h,\alpha)$

$U(p,g,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i' \Delta\alpha_i = \mathrm{U}(p,h,\alpha)$

This is true for every partition P of $[a,b]$

So, $\underline{\int_{a}^{b}} g d\alpha = sup\{L(p,g,\alpha) \mid p \; is \; a \; partition \; of \; [a,b]\}$

$\leq sup\{L(p,g,\alpha) \mid p\} = \underline{\int_{a}^{b}} h d\alpha$

Similarly $\overline{\int_{a}^{b}} g d\alpha \leq \overline{\int_{a}^{b}} h d\alpha$

Claim: $f \in \mathbb{R}(\alpha)$ and $\lim_{n\to\infty} \int_{a}^{b} f_n d\alpha = \int_{a}^{b} f d\alpha$

Let $\varepsilon > 0$ be given

Since $\{f_n\}$ converges uniformly on $[a,b]$ to f, there exists a positive integer $N(\varepsilon)$ such that

$|f_n(x) - f(x)| < \frac{\varepsilon}{2[1+\alpha(b)-\alpha(a)]}$ $whenever \; n \geq N(\varepsilon) \forall x \in [a,b]$

$f(x) - \frac{\varepsilon}{2[1+\alpha(b)-\alpha(a)]} < f_n(x) < f(x) + \frac{\varepsilon}{2[1+\alpha(b)-\alpha(a)]}$ whenever $n \geq N(\varepsilon) \forall x \in [a,b]$

Now by the proof given above, it follows that

$$\underline{\int_{a}^{b}} \left(f - \frac{\varepsilon}{2[1+\alpha(b)-\alpha(a)]}\right) d\alpha \leq \underline{\int_{a}^{b}} f_n d\alpha \leq \underline{\int_{a}^{b}} \left(f + \frac{\varepsilon}{2[1+\alpha(b)-\alpha(a)]}\right) d\alpha \quad \text{for } n \geq N(\varepsilon)$$

$\underline{\int_{a}^{b}} f d\alpha - \varepsilon < \underline{\int_{a}^{b}} f_n d\alpha < \underline{\int_{a}^{b}} f d\alpha + \varepsilon \; for \; n \geq N(\varepsilon)$

$\left|\underline{\int_{a}^{b}} f_n d\alpha - \underline{\int_{a}^{b}} f d\alpha\right| < \varepsilon \; for \; n \geq N(\varepsilon)$

$\therefore \lim_{n\to\infty} \underline{\int_{a}^{b}} f_n d\alpha = \underline{\int_{a}^{b}} f d\alpha$

By symmetry, we have that $\lim_{n\to\infty} \overline{\int_{a}^{b}} f_n d\alpha = \overline{\int_{a}^{b}} f d\alpha$

So, since $\underline{\int_{a}^{b}} f_n d\alpha = \overline{\int_{a}^{b}} f d\alpha$,

$\lim_{n\to\infty} \underline{\int_{a}^{b}} f_n d\alpha = \lim_{n\to\infty} \overline{\int_{a}^{b}} f_n d\alpha$

$\therefore \underline{\int_{a}^{b}} f d\alpha = \lim_{n\to\infty} \underline{\int_{a}^{b}} f_n d\alpha = \lim_{n\to\infty} \overline{\int_{a}^{b}} f_n d\alpha = \overline{\int_{a}^{b}} f d\alpha$

Hence, $f \in \mathbb{R}(\alpha)$ and $\int_{a}^{b} f d\alpha = \lim_{n\to\infty} \int_{a}^{b} f_n d\alpha.$

3.3.2. Theorem: Let α be a bounded variation on $[a,b]$. Assume that each term of the sequence $\{f_n\}$ is a real valued function such that $f_n \epsilon R(\alpha)$ on $[a,b]$ for each $n = 1,2,...$ Assume that $f_n \to f$ uniformly on $[a,b]$ and define $g_n(x) = \int_a^x f_n(t)d\alpha(t)$ if $x\epsilon[a,b], n = 1,2,...$ Then we have:

a) $f\epsilon R(\alpha)$ on $[a,b]$.
b) $g_n \to g$ uniformly on $[a,b]$, where $g(x) = \int_a^x f(t)d\alpha(t)$.

Note: The conclusion implies that, for each x in [a, b], we can write

$\lim_{n\to\infty} \int_a^x f_n(t)d\alpha(t) = \int_a^x \lim_{n\to\infty} f_n(t)d\alpha(t)$.

This property is often described by saying that a uniformly convergent sequence can be integrated term by term.

Proof: we can assume that α is increasing with $\alpha(a) < \alpha(b)$.

To prove (a)

We will show that f satisfies Riemann's condition with respect to α on [a, b].

Given $\varepsilon > 0$, choose N so that

$|f(x) - f_N(x)| < \frac{\varepsilon}{3[\alpha(b)-\alpha(a)]}$, for all x in [a, b].

Then, for every partition P of [a, b], we have

$|U(P, f - f_N, \alpha)| \le \frac{\varepsilon}{3}$ and $|L(P, f - f_N, \alpha)| \le \frac{\varepsilon}{3}$

For this N, choose P_ε so that P finer than P_ε implies

$U(P, f_N, \alpha) - L(P, f_N, \alpha) < \frac{\varepsilon}{3}$.

Then for such P we have

$$U(P, f, \alpha) - L(P, f, \alpha) \le U(P, f - f_N, \alpha) - L(P, f - f_N, \alpha)$$
$$+U(P, f_N, \alpha) - L(P, f_N, \alpha)$$
$$< |U(P, f - f_N, \alpha)| + |L(P, f - f_N, \alpha)| + \frac{\varepsilon}{3} \le \varepsilon.$$

This proves (a).

To prove (b), let $\varepsilon > 0$ be given and choose N so that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{2[\alpha(b) - \alpha(a)]},$$

For all $n > N$ and every t in $[a, b]$. If $x \in [a, b]$, we have

$$|g_n(x) - g(x)| \le \int_a^x |f_n(t) - f(t)| d\alpha(t) \le \frac{\alpha(x)-\alpha(a)}{\alpha(b)-\alpha(a)} \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} < \varepsilon.$$

This proves that $g_n \to g$ uniformly on $[a, b]$.

3.3.3 Theorem: If a series $\sum f_n$ uniformly converges to f on $[a, b]$ and each f_n is continuous on $[a, b]$ then f is integrable on $[a, b]$ and the series $\sum\left(\int_a^x f_n dt\right)$ converges uniformly to $\int_a^x f\, dt$, for all x in $[a, b]$, i.e.,

$$\int_a^x f\, dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right), \text{ for all } x \in [a, b]$$

Proof: Since $\sum f_n$ is uniformly convergent to f on $[a, b]$ and each f_n continuous on $[a, b]$, therefore the sum function f is continuous and hence integrable on $[a, b]$.

Again, since all the functions f_n are continuous, therefore the sum of finite number of functions, $\sum_{r=1}^{n} f_r$ is also continuous and integrable on $[a, b]$, and

$$\sum_{r=1}^{n} \int_a^x f_r dt = \int_a^x \sum_{r=1}^{n} f_r dt$$

By the uniform convergence of the series, for $\varepsilon > 0$, we can find an integer N such that for all x in $[a, b]$

$$|f - \sum_{r=1}^{n} f_r| < \frac{\varepsilon}{(b-a)}, \text{ for all } n \ge N$$

For such values of n, and all x in $[a, b]$

$$\left|\int_a^x f dt - \sum_{r=1}^{n} f_r dt\right| = \left|\int_a^x (f - \sum_{r=1}^{n} f_r) dt\right|$$

$$\le \int_a^x |f - \sum_{r=1}^{n} f_r|\, dt$$

$$< \frac{\varepsilon}{b-a} \int_a^x dt \le \varepsilon$$

That implies $\sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right)$ converges uniformly to $\int_a^x f\, dt$ on $[a, b]$

That is $\int_a^x f\, dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n dt\right)$, for all $x \in [a, b]$

3.4 EXAMPLES:

3.4.1 Example 1: The series

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}, (0 < x < 1)$$

Each term is integrable.

Solution: Given equations is

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}, (0 < x < 1)$$

Integrating from 0 to a, the right hand side gives

$$\int_0^1 \frac{dx}{1+x} = log\ 2$$

While the other side gives

$$\int_0^1 (1 - x + x^2 - x^3 + \cdots) dx = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]_0^1$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

But we know that

$$log\ 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Thus the two sides are equal at $x = 1$, and so term by term integration is possible over [0,1], even though the given series is not uniformly convergent on [0,1].

3.4.2 Example 2: The sequence $\{f_n\}$, where

$$f_n(x) = nxe^{-nx^2}, n = 1, 2, 3, \ldots.$$

Converges point wise to zero on [0,1].

Solution: Given that

$$f_n(x) = nxe^{-nx^2}, n = 1, 2, 3, \ldots.$$

Here

$$\int_0^1 f\ dx = 0$$

and

$\int_0^1 f_n dx = \frac{1}{2}\left[-e^{-nx^2}\right]_0^1 = \frac{1}{2}(1 - e^{-n})$

Therefore

$$\lim_{n\to\infty} \int_0^1 f_n dx = \lim_{n\to\infty} \frac{1}{2}(1 - e^{-n}) = \frac{1}{2} \neq \int_0^1 f\, dx.$$

Hence, convergence cannot be uniform on [0,1].

Note: if we, first, show that the sequence is non-uniformly convergent, then this is an example of a sequence which, though not uniformly convergent yet, has an integrable limit function.

3.4.3 Example 3: Show that the sequence $\{f_n\}$, where

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ -n^2 x + 2n & 1/n \le x \le 2/n \\ 0, & 2/n \le x \le 1 \end{cases}$$

is not uniformly convergent on [0,1].

Solution: Given that

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ -n^2 x + 2n & 1/n \le x \le 2/n \\ 0, & 2/n \le x \le 1 \end{cases}$$

The sequence converges to f, where $f(x) = 0$, for all x belongs to [0,1]. Each function f_n and f are continuous on [0,1].

Also

$$\int_0^1 f_n\, dx = \int_0^{1/n} n^2 x\, dx + \int_{1/n}^{2/n} (-n^2 x + 2n) dx + \int_{2/n}^1 0\, dx = 1$$

But

$$\int_0^1 f\, dx = 0$$

Therefore

$$\lim_{n\to\infty} \int_0^1 f_n dx \neq \int_0^1 f\, dx.$$

So (Theorem 3.2.1) the sequence $\{f_n\}$ cannot convergent uniformly on [0,1].

3.4.4 Example 4: Show that the sequence $\{f_n\}$, where

$f_n(x) = \frac{\log(1+n^3x^2)}{n^2}$

is uniformly convergent on the interval [0,1].

Solution: Given function is

$f_n(x) = \frac{\log(1+n^3x^2)}{n^2}$

The sequence $\{\varphi_n\}$, where $\varphi_n(x) = \frac{2nx}{1+n^3x^2} \equiv f'_n(x)$, may be easily shown to be uniformly convergent to φ, where $\varphi(x) = 0$, on $[0, 1]$. Also each function φ_n is continuous on the given interval.

Therefore (by Theorem 3.2.1) the sequence of its integrals, $\{f_n\}$ converges uniformly to $\int_0^x \varphi \ dt = 0$ on $[0, 1]$.

3.5 SUMMARY:

In this Lesson we are given detailed explanation about uniform convergence and integration of function through definitions and theorems.

3.6 TECHNICAL TERMS:

- Non-uniformly convergent
- Integrable limit function
- Uniformly convergent

3.7 SELF-ASSESSMENT QUESTIONS:

1. If $I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$ if $\{x_n\}$ is a sequence of district points of (a, b), and if $\sum|c_n|$ converges, prove that the series $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$ converges uniformly, and that f is continuous for every $x \neq x_n$.
2. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that $\lim\limits_{n\to\infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?.
3. Let α be bounded variation on $[a, b]$ and assume that $\sum f_n(x) = f(x)$ (uniformly on $[a, b]$), where each f_n is a real-valued function such that $f_n \in R(\alpha)$ on $[a, b]$.

   Then we have

   a) $f \in R(\alpha)$ on $[a, b]$.

b) $\int_a^x \sum_{n=1}^{\infty} f_n(t)d\alpha(t) = \sum_{n=1}^{\infty} \int_a^x f_n(t)d\alpha(t)$ (uniformly on $[a, b]$).

Hint: Apply Theorem 2.2.2 to the sequence of partial sums.

Note: This theorem is described by saying that a uniformly convergent series can be integrated by term by term.

3.8 SUGGESTED READINGS:

1. Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985.
3. Mathematical Analysis by S.C. Malik and Savita Arora, New Age International (P) Limited, 2nd Edition, 1997.

- **Dr. K. Gangadhar**

LESSON - 4
UNIFORM CONVERGENCE AND DIFFERENTIATION

OBJECTIVES:

After studying the lesson you should able to:
Understand the concept of Uniform convergence, concept of Differentiation, solved problems and related Theorems.

STRUCTURE:

4.1 Introduction

4.2 Theorems

4.3 Solved Examples

4.4 Summary

4.5 Technical terms

4.6 Self -Assessment Questions

4.7 Suggested readings

4.1 INTRODUCTION:

In this chapter we learn about the definitions of Uniform convergence, Differentiable, Monotonically increasing, Monotonically decreasing and Riemann Integrable. We also know about some Theorems and Solved problems.

4.1.1 Definition: Uniform convergence

A function $f : D \to R$ is uniformly continuous if for every $\in > 0$, there exist a $\delta > 0$ such that $|f(x) - f(t)| < \in$ for all $x, t \in D$ satisfying $|x - t| < \delta$.

Examples:

1) Linear functions are examples of uniformly continuous functions.
2) Every continuous function on a compact interval is uniformly continuous functions.

4.1.2 Definition: Differentiable

Let $f : [a,b] \to R$ and $x \in [a,b]$. Suppose $a < t < b$ and $t \neq x$. If $\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ exist.

Then it is called derivative of *f* at *x*. We write it by $f'(x)$.

$$\therefore f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

Then we say that *f* is differentiable at *x*.

4.1.3 Definition: Monotonically increasing

Let *f* be a real valued function on (*a*, *b*). Then *f* is said to be Monotonically increasing, if $a < x < y < b \Rightarrow f(x) \leq f(y)$.

(or)

Suppose f is differentiable function on (a, b). If $f'(x) \geq 0, \forall x \in (a,b)$ then we say that f is Monotonically increasing.

4.1.4 Definition: Monotonically decreasing

Let f be a real valued function on (a, b). Then f is said to be Monotonically decreasing, if $a < x < y < b \Rightarrow f(x) \geq f(y)$.

(or)

Suppose f is differentiable function on (a, b). If $f'(x) \leq 0, \forall x \in (a,b)$ then we say that f is Monotonically decreasing.

4.1.5 Definition: Compact set

Let X be a metric space and $E \subset X$. Then the E is said to be compact, if every open cover for E has a finite sub cover for E.

$\therefore E \subset \bigcup_{\alpha=1}^{\infty} G_\alpha \Rightarrow E \subset G_{\alpha_1} \cup G_{\alpha_2} \cup ...... \cup G_{\alpha_n}$. Here $\{G_\alpha\}$ is a collection of open sets.

4.1.6 Definition: Riemann Integrable

Let f be a bounded function on $[a, b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ be the partition of $[a, b]$, then

$m_i = Inf\{f(x) / x \in [x_{i-1}, x_i]\}, 1 \leq i \leq n$,

$M_i = Sup\{f(x) / x \in [x_{i-1}, x_i]\}, 1 \leq i \leq n$,

$\Delta x_i = x_i - x_{i-1}$.

$\therefore L(P, f) = \sum_{i=1}^{n} m_i \, \Delta x_i$ is called Lower Riemann sum and

$U(P, f) = \sum_{i=1}^{n} M_i \, \Delta x_i$ is called Upper Riemann sum.

Also $\underline{\int_a^b} f\,dx$ = Sup $\{L(P, f) / P$ is a partition of $[a, b]\}$ is called Lower Riemann integral and

$\overline{\int_a^b} f\,dx$ = Inf $\{U(P, f) / P$ is a partition of $[a, b]\}$ is called Upper Riemann integral.

If $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$ then we say that f is Riemann integrable over [a, b]. It is denoted by $\int_a^b f\,dx$.

$$\therefore \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = \int_a^b f\,dx.$$

4.2 THEOREMS:

4.2.1 Theorem: Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$ $(a \le x \le b)$.

Proof:

Let $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ and $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$.

$\Rightarrow$ For given $\in > 0$ there exist a positive integer N_1 such that $|f_n(x_0) - f_m(x_0)| < \frac{\in}{2}$ $\forall\, n, m \ge N_1$.

Let $\{f_n'\}$ converges uniformly on $[a, b]$.

$\Rightarrow$ For given $\in > 0$ there exist a positive integer N_2 such that $|f_n'(t) - f_m'(t)| < \frac{\in}{2(b-a)}$,

$\forall\, n, m \ge N_2$ and $\forall t \in [a,b]$.

Let N = Max $\{N_1, N_2\}$ then $|f_n(x_0) - f_m(x_0)| < \frac{\in}{2}$ and $|f_n'(t) - f_m'(t)| < \frac{\in}{2(b-a)}$, $\forall\, n, m \ge N$ and

$\forall t \in [a,b]$. $\longrightarrow (1)$

By applying the Lagrange's Mean value Theorem to the function $(f_n - f_m)$,

We have for $x, t \in [a,b]$, $|(f_n - f_m)(x) - (f_n - f_m)(t)| = |x-t|.|(f_n - f_m)'(p)|$ for some point p between x and t, if $n, m \ge N$.

For any $x, t \in [a,b]$, we have

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| < |x-t|.\frac{\in}{2(b-a)} \quad (\because by (1))$$

$$\le \frac{\in}{2} \quad \left(\because \frac{|x-t|}{(b-a)} \le 1\right) \qquad \forall\, n, m \ge N.$$

$$\therefore |f_n(x) - f_m(x) - f_n(t) + f_m(t)| < \frac{\in}{2} \quad --\to (2)$$

For any $x \in$ [a, b] and $n, m \ge N$,

We have $|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)|$

$$\le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

$$\le \frac{\in}{2} + \frac{\in}{2} \quad (\because by (1) and (2))$$

$$= \in$$

$$\therefore |f_n(x) - f_m(x)| < \in, \forall n, m \ge N, x \in [a,b].$$

$\Rightarrow$ $\{f_n\}$ converges uniformly on $[a, b]$.

Claim: $f'(x) = \lim_{n\to\infty} f_n'(x) \quad \forall x \in [a,b]$.

Since $\{f_n\}$ converges uniformly to a function f on $[a, b]$.

$\Rightarrow \lim_{n\to\infty} f_n(x) = f(x)$

Fix $x \in [a, b]$.

For any $a \le t \le b$ with $t \ne x$, define $\psi(t) = \dfrac{f(t) - f(x)}{t - x}$ and $\psi_n(t) = \dfrac{f_n(t) - f_n(x)}{t - x}$.

Now $\lim_{t\to x} \psi_n(t) = \lim_{t\to x} \dfrac{f_n(t) - f_n(x)}{t - x} = f_n'(x), a \le x \le b, n = 1,2,3....$ $\qquad --\to (3)$

From (1) we have $|f_n'(t) - f_m'(t)| < \dfrac{\in}{2(b-a)}, \ \forall\ n, m \ge N,\ \forall t \in [a,b]$

$\Rightarrow \left|\lim_{t\to x} \psi_n(t) - \lim_{t\to x} \psi_m(t)\right| \le \dfrac{\in}{2(b-a)},\ (\because by (3))$

$\Rightarrow |\psi_n(t) - \psi_m(t)| \le \dfrac{\in}{2(b-a)}, \forall\ n, m \ge N,\ \forall t \in [a,b]$.

Therefore by Cauchy - Criterion for uniform convergence we have $\{\psi_n(x)\}$ converges uniformly for $t \ne x$.
Since $\{f_n\}$ converges to f, we conclude that

$\lim_{n\to\infty} \psi_n(t) = \lim_{n\to\infty} \dfrac{f_n(t) - f_n(x)}{t - x} = \dfrac{f(t) - f(x)}{t - x} = \psi(t)$ Uniformly for $a \le t \le b$

with $t \ne x$. $\qquad —\to (4)$

Then by known theorem, (3) and (4) shows that $\lim_{t\to x} \psi(t) = \lim_{n\to\infty} f_n'(x)$.

$\therefore \lim_{n\to\infty} f_n'(x) = \lim_{t\to x} \psi(t) = \lim_{t\to x} \dfrac{f(t) - f(x)}{t - x} = f'(x)$, for $a \le x \le b$.

$\Rightarrow \lim_{n\to\infty} f_n'(x) = f'(x) \quad \forall x \in [a,b]$.

4.2.2 Theorem: There exists a real continuous function on the real line which is nowhere differentiable.

Proof:
Define $\phi(x) = |x|$ where $-1 \le x \le 1$, and

We extend ϕ to R by $\phi(x+2) = \phi(x)$ in the following.

We know that every real number must be in an interval of the form $[2n-1, 2n+1]$, for some integer n.
Define $\phi : R \to R$ by $\phi(x) = |x - 2n|$ if $2n-1 \le x \le 2n+1$.

Then $\phi(m) = \begin{cases} 1, & if\ m\ is\ an\ even\ integre. \\ 0, & if\ m\ is\ an\ odd\ integer. \end{cases}$

$\therefore\ \phi(x) = |x|$, for -1≤ x ≤ 1 and also $\phi(x+2n) = \phi(x)$ for all $x \in R, n \in Z$.

Let $m \in Z$,

We have $2m-1 \le x \le 2m+1$

$\Rightarrow\ (2m-1)+2n \le x+2n \le (2m+1)+2n.$

$\Rightarrow 2(m+n)-1 \le x+2n \le 2(m+n)+1.$

$\therefore \phi(x+2n) = |(x+2n)-2(m+n)| = |x-2m| = \phi(x).$

Since 2 is the least positive period, so ϕ is periodic function with period 2.

Clearly ϕ is continuous on R .

For $x \in R$, Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi\left(4^n x\right).$

Since $0 \le \phi(x) \le 1$ so $\left|\left(\frac{3}{4}\right)^n \phi\left(4^n x\right)\right| \le \left(\frac{3}{4}\right)^n$.

Since $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$ is convergent then by weierstrass M-Test, the series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n x)$ converges uniformly on R.

Since f is the uniform limit of sequence of continuous functions it follows that f is continuous on R.

Prove that f is differentiable at nowhere.

Let $x \in R$ be a fixed real number and m is any positive integer.

Define $\delta_m = \pm \frac{1}{2.4^m}$

$\Rightarrow |\delta_m| = \left|\pm \frac{1}{2.4^m}\right| = \frac{1}{2.4^m}$

$\Rightarrow 4^m |\delta_m| = \frac{1}{2}$ is not an integer.

Define $\gamma_n = \frac{\phi\left(4^n(x+\delta_m)\right) - \phi\left(4^n(x)\right)}{\delta_m}$.

If $n > m$ then $n = m + p$, for some positive integer p.

To prove $\gamma_n = 0$ and $4^n.\delta_m$ is an even integer.

Consider $4^n.\delta_m = 4^n.\left(\pm\frac{1}{2.4^m}\right)$

$$= \pm \frac{1}{2}.4^{n}.4^{-m}$$
$$= \pm \frac{1}{2}.4^{n-m}$$
$$= \pm \frac{1}{2}.4^{m+p-m} \; (\because n = m + p)$$
$$= \pm \frac{1}{2}.4^{p}$$
$$= \pm 2^{2p-1}$$
$$= \pm 2k \text{ is an even integer.}$$

$\therefore\ 4^n.\delta_m$ is an even integer.

Consider $\gamma_n = \dfrac{\phi\left(4^n(x+\delta_m)\right) - \phi\left(4^n(x)\right)}{\delta_m}$

$$= \frac{\phi\left(4^n x + 4^n \delta_m\right) - \phi\left(4^n(x)\right)}{\delta_m}$$
$$= \frac{\phi\left(4^n x\right) - \phi\left(4^n(x)\right)}{\delta_m} \; \left(\because \phi(x+2n) = \phi(x)\right)$$
$$= 0$$

$\therefore\ \gamma_n = 0$ if $n > m$. $\longrightarrow (1)$

Show that $|\gamma_n| \le 4^n$ if $n \le m$.

Since $\gamma_n = \dfrac{\phi\left(4^n(x+\delta_m)\right) - \phi\left(4^n(x)\right)}{\delta_m}$

$$\Rightarrow |\gamma_n| = \frac{\left|\phi\left(4^n(x+\delta_m)\right) - \phi\left(4^n(x)\right)\right|}{|\delta_m|}$$
$$= \frac{\left|\phi\left(4^n(x) + 4^n\delta_m\right) - \phi\left(4^n(x)\right)\right|}{|\delta_m|}$$
$$= \frac{\left|\left|4^n(x) + 4^n\delta_m\right| - \left|4^n(x)\right|\right|}{|\delta_m|} \; \left(\because \phi(x) = |x|\right)$$
$$\le \frac{\left|\left|4^n(x)\right| + \left|4^n\delta_m\right| - \left|4^n(x)\right|\right|}{|\delta_m|}$$
$$= \frac{\left|4^n\delta_m\right|}{|\delta_m|}$$
$$= \left|\frac{4^n\delta_m}{\delta_m}\right| = 4^n$$

$\therefore |\gamma_n| \le 4^n$ if $n \le m$. $\longrightarrow$ (2)

Now $$\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| = \left|\frac{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi\left(4^n(x+\delta_m)\right) - \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \phi(4^n x)}{\delta_m}\right|$$

$$\le \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \left|\frac{\phi\left(4^n(x+\delta_m)\right)-\phi(4^n x)}{\delta_m}\right|$$

$$= \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n |\gamma_n| \ (\because \textit{by definition of } \gamma_n)$$

$$= \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n |\gamma_n| + \sum_{n=m+1}^{\infty}\left(\frac{3}{4}\right)^n |\gamma_n|$$

$$= \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n |\gamma_n| + 0 \ \ (\because \textit{by}\ (1)\ \textit{for}\ n > m)$$

$$\le \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n 4^n \ \ (\because \textit{by}\ (2)\ \textit{for}\ n \le m)$$

$$= \sum_{n=0}^{m}\frac{3^n}{4^n}4^n$$

$$= \sum_{n=0}^{m}3^n$$

$$\therefore \left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \le \sum_{n=0}^{m}3^n$$

$$\Rightarrow \sum_{n=0}^{m}3^n \ge \left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| \ge 3^m - \sum_{n=0}^{m-1}3^n$$

Take as $m \to \infty$ then $\delta_m \to 0$.

$\therefore f$ is not differentiable, because $3^m - \sum_{n=0}^{m-1}3^n$ is large value as $m \to \infty$.

$\therefore$ The function f is continuous at everywhere on R but nowhere differentiable on R.

4.3 SOLVED EXAMPLES:

4.3.1 Example: Give an example the limit of integral need not be equal to the Integral of limit.

Solution:

Consider $f_n(x) = nx(1-x^2)^n, 0 \le x \le 1, n = 1, 2, 3, ...$

Claim: $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1\left[\lim_{n\to\infty} f_n(x)\right]dx$.

Now $\int_0^1 f_n(x)dx = \int_0^1 nx(1-x^2)^n\,dx = \frac{n}{-2}\int_0^1 -2x(1-x^2)^n\,dx$

$$= \frac{n}{-2}\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1 = \frac{-n}{2}\left[0-\frac{1-0}{n+1}\right] = \frac{n}{2(n+1)} = \frac{n}{2n(1+\frac{1}{n})} = \frac{1}{2(1+\frac{1}{n})}$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\frac{1}{2(1+\frac{1}{n})} = \frac{1}{2(1+0)} = \frac{1}{2}. \longrightarrow (1)$$

Now $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n = \infty$.

$$\therefore \int_0^1\left[\lim_{n\to\infty} f_n(x)\right]dx = \infty \longrightarrow (2)$$

From (1) and (2) we have $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1\left[\lim_{n\to\infty} f_n(x)\right]dx$.

4.3.2 Example: If *f* is real function and $\lim_{h\to 0}\left[f(x+h)-f(x-h)\right]=0$. Thus this implies that *f* is continuous on *R*.

Solution:

Consider $f:R\to R$ be a function defined by $\begin{aligned} f(x) &= |x| \quad if\ x\neq 0 \\ &= 1 \quad if\ x=0 \end{aligned}$.

Since $\lim_{x\to 0} f(x) = \lim_{x\to 0}|x| = 0$ and $f(0) = 1$.

$\therefore \lim_{x\to 0} f(x) \neq f(0)$ so *f* is not continuous at $x = 0$.

If $x = 0$ then $\lim_{h\to 0}\left[f(x+h)-f(x-h)\right]$

$$\begin{aligned} &= \lim_{h\to 0}\left[f(h)-f(-h)\right] \\ &= \lim_{h\to 0}\left[|h|-|-h|\right] \\ &= \lim_{h\to 0}\left[|h|-|h|\right] \\ &= 0 \end{aligned}$$

$\therefore \lim_{h\to 0}\left[f(x+h)-f(x-h)\right]=0$ does not implies that *f* is continuous on *R*.

4.4 SUMMARY:

Uniform convergence allows us to interchange limits and differentiation under certain if a sequence of differentiable functions converges uniformly and the derivatives also conditions. That means converges uniformly. Also the limit function is differentiable and its derivative is the limit of the derivatives.

4.5 TECHNICAL TERMS:

- Differentiable
- Monotonically increasing
- Monotonically decreasing
- Compact set
- Riemann integrable

4.6 SELF- ASSESSMENT QUESTIONS:

1. For n = 1, 2, 3,.., x real, put $f_n(x) = \dfrac{x}{1+nx^2}$. Show that $\{f_n\}$ converges uniformly to a function f, and that the equation $f'(x) = \lim_{n\to\infty} f'_n(x)$ is correct if $x \neq 0$, but false if $x = 0$.

2. If $I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$ if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series $f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$ $(a \leq x \leq b)$ converges uniformly, and that f is continuous for every $x \neq x_n$.

3. Let $\{f_n\}$be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$,and $x \in E$. Is the converse of this true?

4. Letting $g(x)$ denotes the fractional part of the real number x, consider the function $f(x) = \sum_{n=1}^{\infty} \dfrac{(nx)}{n^2}$ (x real). Find all discontinuities of f, and show that they from a countable dense set. Show that f is nevertheless Riemann-integral on every bounded interval.

5. Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

   (a) $\sum f_n$ has uniformly bounded partial sums;

   (b) $g_n \to 0$ Uniformly on E;

   (c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq ....$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E.

6. Suppose g and f_n (n = 1, 2, 3,…) are defined on (0, ∞), are Rieman- integrable on

$[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \le g, f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and $\int_0^\infty g(x)dx < \infty.$ Prove that $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.$

7. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on R^1 with $0 \le f_n(x) \le 1$ for all x and all n.

(a) Prove that there is a function f and a sequence such that $f(x) = \lim_{k\to\infty} f_{nk}(x)$ for every $x \in R^1$.

(b) If, moreover, f is continuous, prove that $f_{nk} \to f$ uniformly on R^1.

8. Let f be a continuous real function on R^1 with the following properties: for every t,

and $f(t) = \begin{cases} 0 \ \ (0 \le t \le \frac{1}{3}) \\ 1 \ (\frac{2}{3} \le t \le 1). \end{cases}$

Put $\Phi(t) = (x(t), y(t))$, where $x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$ Prove that Φ is continuous and that Φ maps $I = [0, 1]$ onto the unit square $I^2 \subset R^2$. If fact, show that Φ maps the Cantor set onto I^2.

4.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M. Apostal, Narosa Publishing House, 2nd Edition, 1985.

- Dr. N.S.L.V. Narasimharao.

LESSON - 5
EQUICONTINUOUS FAMILY OF FUNCTIONS

OBJECTIVES:

After studying the lesson you should able to:

Illustrate the effects of Equicontinuous, Uniform bounded Family of functions and Uniformly convergent subsequence.

STRUCTURE:

5.1 INTRODUCTION:

In this chapter we learn about the definitions of Point Wise Bounded Sequence, Uniform Bounded Sequence, Equicontinuous and Riemann Integrable. We also know about some Theorems and Solved problems.

5.1.1 Definition: Point Wise Bounded Sequence

Let E be a subset of a Metric space 'X' and let $\{f_n\}$ be a sequence of functions defined on E. We say that $\{f_n\}$ is point wise bounded on E, if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$ if there exists a finite valued function ϕ defined on E such that $|f_n(x)| < \phi(x)\ \forall x \in E, n = 1,2,3,....$

5.1.2 Definition: Uniform Bounded Sequence

Let E be a subset of a Metric space 'X' and let $\{f_n\}$ be a sequence of functions defined on E. We say that $\{f_n\}$ is Uniform bounded on E, if there exists a number M such that $|f_n(x)| < M\ \forall x \in E, n = 1,2,3,....$

5.1.3 Definition: Equicontinuous

A family '*F*' of complex functions defined on E in a metric space (*X*, *d*) is said to be Equicontinuous on *E* if for every ε>0 there exists a δ>0 such that $|f(x)-f(y)|<\varepsilon$ whenever $d(x, y) < \delta$ for $x, y \in E$ and $f \in F$.

5.1.4 Definition: Riemann Integrable

Let *f* be a bounded function on [*a*, *b*] and P = {x_0, x_1, …,x_n} be the partition of [*a*, *b*],

then $m_i = Inf\{f(x)/x \in [x_{i-1}, x_i]\}, 1 \le i \le n$,

$M_i = Sup\{f(x)/x \in [x_{i-1}, x_i]\}, 1 \le i \le n$,

$\Delta x_i = x_i - x_{i-1}$.

$\therefore L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$ is called Lower Riemann sum and

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ is called Upper Riemann sum.

Also $\underline{\int_a^b} f\,dx$ = Sup {*L*(*P*, *f*) / *P* is a partition of [*a*, *b*] } is called Lower Riemann integral and

$\overline{\int_a^b} f\,dx$ = Inf {*U*(*P*, *f*) / *P* is a partition of [*a*, *b*] } is called Upper Riemann integral.

If $\underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx$ then we say that *f* is Riemann integrable over [a, b]. It is denoted by $\int_a^b f\,dx$.

5.2 SOLVED EXAMPLES:

5.2.1 Example: If $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}, x \in [0,1], n = 1,2,3,...$ then show that $\{f_n\}$ is Uniformly bounded on [0,1] and It has no subsequence which converges uniformly on [0,1].

(or)

The sequence $\{f_n\}$ converges on [0,1], but not uniformly.

Solution:

Let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2}, x \in [0,1], n = 1,2,3,...$

Since $(1-nx)^2 \ge 0$

$\Rightarrow x^2 + (1-nx)^2 \geq 0 + x^2$

$\Rightarrow \dfrac{x^2}{x^2 + (1-nx)^2} \leq 1$

$\Rightarrow f_n(x) \leq 1, \ \forall x \in [0,1], n = 1,2,3,...$

$\therefore \{f_n\}$ is uniformly bounded on [0,1].

Now $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x^2}{x^2 + (1-nx)^2} = \dfrac{x^2}{x^2 + \infty} = 0$.

$\therefore \{f_n\}$ converges to '0'.

Let positive integer 'n' there exist $\dfrac{1}{n} \in [0,1]$.

Now $f_n(1/n) = \dfrac{(1/n)^2}{(1/n)^2 + (1-n.(1/n))^2} = \dfrac{(1/n)^2}{(1/n)^2 + 0} = 1$

$\Rightarrow \{f_n(1/n)\}$ converges to 1.

$\therefore \{f_n\}$ has no subsequence which converges uniformly on[0,1].

5.2.2 Example: If $f_n(x)$ = sin nx, $0 \leq x \leq 2n$, n = 1, 2, 3,….then there is no subsequence which converges point wisely on $[0, 2\pi]$.

Solution:

Let $f_n(x)$ = sin nx, $0 \leq x \leq 2n$, n = 1,2,3,….

Now $|f_n(x)| = |\sin nx| \leq 1$

$\Rightarrow |f_n(x)| \leq 1 \forall x \in [0, 2\pi], \forall n$

$\therefore \{f_n\}$ is uniformly bounded sequence of continuous functions on a compact set $[0, 2\pi]$.

In contrary way, we suppose that a subsequence $\{f_{nk}\}$ of converges point wisely on $[0, 2\pi]$.

Now $\lim\limits_{k\to\infty} (\sin n_k x - \sin n_{k+1} x) = 0 - 0 = 0$.

The sub sequence $\{f_{nk}\}$ converges to '0'.

Since $\lim\limits_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0$

By Lebesgue Integral formula on bounded convergent sequence, we have

$\lim\limits_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 = 0$ so $2\pi = 0$.It is a contradiction.

Hence there is no subsequence $\{f_{nk}\}$ which converges point wisely on $[0, 2\pi]$.

5.3 THEOREMS:

5.3.1 Theorem: If $\{f_n\}$ is a point wise bounded sequence of complex functions on a countable set *E*, then $\{f_n\}$ has a sub sequence $\{f_{nk}\}$ such that $\{f_{nk}(x)\}$ converges for every $x \in E$.

Proof:

Let $\{f_n\}$ is a point wise bounded sequence of complex functions on a countable set *E*.

Since *E* is Countable set then the elements of *E* can be arranged distinct sequence.

Let $E = \{x_1, x_2, \ldots\}$, where $x_i \neq x_j$, for $i \neq j$.

Since $\{f_n\}$ is a point wise bounded on *E*.

$\Rightarrow \{f_n(x_i)\}$ is bounded for all $x_i \in E$.

Since $\{f_n(x_1)\}$ is bounded then by known theorem (since every bounded sequence have convergence sub sequence) $\{f_n(x_1)\}$ have a convergent subsequence $\{f_{1k}(x_1)\}$ say.

Let $S_1 = \{f_{11}, f_{12}, f_{13}, \ldots..\}$

Since $\{f_n(x_2)\}$ is bounded then by known theorem (since every bounded sequence have convergence sub sequence) $\{f_n(x_2)\}$ have a convergent subsequence $\{f_{2k}(x_2)\}$ say.

Let $S_2 = \{f_{21}, f_{22}, f_{23}, \ldots..\}$

Continuing this process we get

$S_1 = \{f_{11}, f_{12}, f_{13}, \ldots..\}$

$S_2 = \{f_{21}, f_{22}, f_{23}, \ldots..\}$

$S_3 = \{f_{31}, f_{32}, f_{33}, \ldots..\}$

....

From above argument we have

(i) S_n is a sub sequence of S_{n-1}, for n = 2,3,4….

(ii) $\{f_{nk}(x_n)\}$ Converges.

Consider the diagonal elements $S = \{f_{11}, f_{22}, f_{33}, \ldots..\}$

Clearly *S* is sub sequence S_n for *n* = 1, 2, 3,…. and $\{f_{nn}(x_i)\}$ is converges. $\forall x_i \in E$

Hence $\{f_n\}$ has a sub sequence $\{f_{nn}\}$ such that $\{f_{nn}(x_i)\}$ converges for every $x_i \in E$.

5.3.2 Theorem: If K is compact metric space. If $f_n \in \mathbb{C}(K)$ for n = 1, 2, 3,…. and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is Equicontinuous on K.

Proof:

Given K is compact metric space.

Given $f_n \in \mathbb{C}(K)$ for n = 1, 2, 3,….

$\Rightarrow f_n$ is complex valued continuous bounded function on K, for n = 1,2,3,….

Since f_n is continuous function on K and K is compact

$\Rightarrow$ f_n is Uniformly continuous on K.

$\Rightarrow$ For given $\in > 0 \; \exists \delta > 0 \ni |f_n(x) - f_n(y)| < \frac{\in}{3}$ whenever $d(x,y) < \delta, \forall x, y \in K$,

for n = 1, 2, 3,… $--\rightarrow$ (1)

Since $\{f_n\}$ is Converges Uniformly on K.

$\Rightarrow$ For given $\in > 0, \exists$ a positive integer $N \ni |f_n(x) - f_m(x)| < \frac{\in}{3}, \; \forall n, m \geq N, \forall x \in K$. $--\rightarrow$ (2)

$\Rightarrow |f_n(x) - f_N(x)| < \frac{\in}{3}, \; \forall n \geq N, \forall x \in K$. $--\rightarrow$ (3)

Claim: $\{f_n\}$ is Equicontinuous on K.

Let $d(x,y) < \delta, \forall x, y \in K$.

Consider $|f_n(x) - f_n(y)| = |f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)|$

$$\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} \quad (By (1),(2),(3))$$

$$= \in$$

Hence for given $\in > 0 \; \exists \delta > 0 \ni |f_n(x) - f_n(y)| < \in$, whenever $d(x,y) < \delta, \forall x, y \in K$,

for n = 1,2,3,…

$\Rightarrow \{f_n\}$ is Equicontinuous on K.

5.3.3 Theorem: If K is compact. If $f_n \in \mathbb{C}(K)$ for n = 1, 2, 3,…. and $\{f_n\}$ is point wise bounded and Equicontinuous on K, then (a) $\{f_n\}$ is Uniform bounded on K.

(b) $\{f_n\}$ contains Uniformly convergent subsequence.

Proof:

Given K is compact and $f_n \in \mathbb{C}(K)$ for $n = 1,2,3,\ldots$

Given $\{f_n\}$ is point wise bounded and Equicontinuous on K.

Proof (a):

Since $\{f_n\}$ is Equicontinuous on K.

$\Rightarrow$ For given $\in > 0 \exists \delta > 0 \ni |f_n(x) - f_n(y)| < \in$ when ever $d(x,y) < \delta, \forall x, y \in K$,

for $n = 1,2,3,\ldots$ $\quad --\rightarrow (1)$

Since K is compact set

$\Rightarrow$ There exist finite points $p_1, p_2, \ldots. p_r \in K \ni K \subseteq N_\delta(p_1) \cup N_\delta(p_2) \cup \ldots\ldots \cup N_\delta(p_r)$

Since $\{f_n\}$ is point wise bounded on K

$\Rightarrow \{f_n(p_i)\}$ is bounded, for $i = 1,2,3,\ldots..$

$\Rightarrow$ There exist positive integer $M_i \ni |f_n(p_i)| \le M_i$, for $i = 1,2,3,\ldots..$ $\quad --\rightarrow (2)$

Let $M = \text{Max } \{M_1, M_2, \ldots\ldots M_r\}$

$\Rightarrow M_i \le M$, for $i = 1,2,3,\ldots..$ $\quad --\rightarrow (3)$

Let $x \in K \Rightarrow x \in N_\delta(p_1) \cup N_\delta(p_2) \cup \ldots\ldots \cup N_\delta(p_r)$

$\Rightarrow x \in N_\delta(p_i)$, for some $i = 1,2,3,\ldots..$

$\Rightarrow d(x, p_i) < \delta$

$\Rightarrow |f_n(x) - f_n(p_i)| < \in$ (since by (1))

$\Rightarrow |f_n(x)| - |f_n(p_i)| < \in \left(\because |x - y| \ge \big||x| - |y|\big|\right)$

$\Rightarrow |f_n(x)| < \in + |f_n(p_i)|$

$\Rightarrow |f_n(x)| \le \in + M_i$ $(\because by (2))$

$\Rightarrow |f_n(x)| \le \in + M$ $(\because by (3))$

$\therefore |f_n(x)| \le \in + M \quad \forall x \in K$, for $n = 1,2,3,\ldots\ldots$

$\Rightarrow \{f_n\}$ is Uniform bounded on K.

Proof (b):

Let E be the countable dense sub set of K. i.e) $\overline{E} = K$.

Then by known theorem (Theorem 1) we have $\{f_n\}$ has sub sequence $\{f_{ni}\}$ such that $\{f_{ni}(x)\}$ convergent, for every $x \in E$.

$\Rightarrow \{f_n\}$ has sub sequence $\{g_i\}$ such that $\{g_i(x)\}$ convergent, for every $x \in E$, where $g_i = f_{ni}$.

Claim: $\{g_i\}$ is uniformly convergent on K.

Since K is compact set

$\Rightarrow$ There exist finite points $x_1, x_2, ...x_r \in E \ni K \subseteq N_\delta(x_1) \cup N_\delta(x_2) \cup ...... \cup N_\delta(x_r)$

Since $\{g_i(x)\}$ convergent for every $x \in E$

$\Rightarrow \{g_i(x_s)\}$ convergent for every $x_s \in E$

$\Rightarrow$ For given $\in > 0, \exists$ a positive integer N $\ni \left|g_i(x_s) - g_j(x_s)\right| < \frac{\in}{3}, \forall i, j \geq N \quad --\rightarrow (4)$

Since $f_n \in \mathbb{C}(K)$

$\Rightarrow f_n$ is continuous function on K.

$\Rightarrow g_i$ is continuous function on K.(since $\{g_i\}$ is a sub sequence of $\{f_n\}$)

If $x \in K$

$\Rightarrow x \in N_\delta(x_1) \cup N_\delta(x_2) \cup ...... \cup N_\delta(x_r)$

$\Rightarrow x \in N_\delta(x_s)\ for\ some\ 1 \leq s \leq r$

$\Rightarrow d(x, x_s) < \delta$

$\Rightarrow \left|g_i(x) - g_i(x_s)\right| < \frac{\in}{3}\ (\because g_i is continuous\ on\ K) --\rightarrow (5)$

Consider

$$\left|g_i(x) - g_j(x)\right| = \left|g_i(x) - g_i(x_s) + g_i(x_s) - g_j(x_s) + g_j(x_s) - g_j(x)\right|$$

$$\left|g_i(x) - g_j(x)\right| \leq \left|g_i(x) - g_i(x_s)\right| + \left|g_i(x_s) - g_j(x_s)\right| + \left|g_j(x_s) - g_j(x)\right|$$

$$\left|g_i(x) - g_j(x)\right| < \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} \qquad (\because by (4), (5))$$

$$\left|g_i(x) - g_j(x)\right| = \in$$

$$\therefore \left|g_i(x) - g_j(x)\right| < \in$$

Hence for given $\in > 0, \exists$ a positive integer N $\ni \left|g_i(x) - g_j(x)\right| < \in, \forall i, j \geq N, \forall x \in K$

$\Rightarrow \{g_i\}$ is Uniformly convergent on K.

5.3.4 Theorem: Prove that a subset S of $\mathbb{C}(K)$ is compact if and only if it is uniformly closed, pointwise bounded, and equicontinuous, where K is a compact metric space.

Proof:

Given K is a compact metric space.

Given S is a subset of $\mathbb{C}(K)$, the set of continuous functions on K.

Necessary part:

Let S is compact.

Claim: S is uniformly closed, pointwise bounded and equicontinuous.

Since S is compact, then it is closed.

Thus, S is uniformly closed because the convergence is uniform.

Since S is compact, it is totally bounded.

Thus, for any $x \in K$, the set $\{f(x) : f \in S\}$ is bounded.

Hence, S is pointwise bounded.

Since S is compact, it is totally bounded. For each $\varepsilon > 0$ there exist a finite set $\{f_1,...,f_n\} \subset S$ such that for any $f \in S$, there exist f_i with $d(f, f_i) < \frac{\varepsilon}{3}$

Each f_i is uniformly continuous, so there exists $\delta > 0$ such that $d(x, y) > \delta$ implies $|f_i(x) - f_i(y)| < \frac{\varepsilon}{3}$, for all i.

For any $f \in S$, choose f_i such that $d(f, f_i) < \frac{\varepsilon}{3}$.

Then

$$|f(x) - f(y)| \le |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Hence, S is equicontinuous.

Sufficient part:

Let S is uniformly closed, pointwise bounded and equicontinuous.

Claim: S is compact.

By the Arzela-Ascoli Theorem, since S is pointwise bounded and equicontinuous, S is relatively compact.

Since S is uniformly closed, it contains all its limit points, and thus is compact.

5.4 SUMMARY:

An Equicontinuous family of functions exhibit similar changes in output when small changes in input, ensuring a uniform approach to convergence and compactness.

5.5 TECHNICAL TERMS:

Point Wise Bounded Sequence

Uniform Bounded Sequence

Equicontinuous

Riemann Integrable

5.6 SELF ASSESSMENT QUESTIONS:

1. Suppose f is a real continuous function on R^1, $f_n(t) = f(nt)$ for $n = 1,2, 3,\ldots$ and $\{ f_n\}$ is equicontinuous on [0,1]. What conclusion can you draw about f ?
2. Suppose $\{f_n\}$ is an equicontinuous family of functions on a compact set K, and $\{f_n\}$ converges point wise on K. Prove that $\{f_n\}$ converges uniformly on K.
3. Let $\{f_n\}$ be a uniform bounded sequence of functions which are Riemann-integrable on [a, b], and put $F_n(x) = \int_a^x f_n(t)dt \quad (a \le x \le b)$. Prove that there exists a subsequence $\{F_{nk}\}$ which converges uniformly on [a, b].
4. Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is a compact if and only if S is uniformly closed, point wise bounded, and equicontinuous.

5.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M. Apostal, Narosa Publishing House, 2nd Edition, 1985.

- **Dr. N.S.L.V. Narasimharao.**

LESSON - 6
STONE – WEIERSTRASS THEOREM

OBJECTIVES:

After studying the lesson you should able to:
Illustrate about Stone – Weierstrass Theorem and its applications.

STRUCTURE:

6.1 INTRODUCTION:

In this lesson we know about the definitions of Sequences, complex valued functions, compact set, continuous and uniform continuous. We also learn about Stone – Weierstrass Theorem and its corollary. We also learn some solved examples.

6.1.1 Definition: Sequence

A function f defined on the set J of all positive integers is said to be a Sequence.
If $f(n) = x_n$, for $n \in J$.

We denote the sequence f by the symbol $\{x_n\}$. The value of f that is, the element x_n , are called the terms of the sequence.

If A is a set and if $x_n \in A$ for all $n \in J$, then is said to be a sequence in A (or) a sequence of elements of A.

6.1.2 Definition: Complex Valued Function

A complex valed function on the interval $[a, b]$ is a function that takes real numbers from the interval $[a, b]$ and maps them to complex numbers. This can be expressed as a function $f: [a, b] \to C$, where C represents the set of all complex numbers.

6.1.3 Definition: Continuous Function

A function $f: D \to R$ is continuous at $x \in D$, if for every $\in > 0$, there exist a $\delta > 0$ such that $|f(x) - f(t)| < \in$ satisfying $|x - t| < \delta$.

6.1.4 Definition: Uniformly Convergent

A sequence of functions $\{f_n(x)\}$ converges uniformly to a function f(x) on a set E, if given $\in > 0$, there exists an integer N such that $|f_n(x) - f(x)| < \varepsilon$, for all $n \geq N$ and all $x \in E$.

6.1.5 Definition: Compact Set

Let X be a metric space and $E \subset X$. Then the E is said to be compact, if every open cover for E has a finite sub cover for E.

$\therefore E \subset \bigcup_{\alpha=1}^{\infty} G_\alpha \Rightarrow E \subset G_{\alpha_1} \cup G_{\alpha_2} \cup ...... \cup G_{\alpha_n}$. Here $\{G_\alpha\}$ is a collection of open sets.

6.1.6 Definition: Bounded function

A bounded function is a function whose out put values are contained within a finite range, meaning they have both a finite upper and lower bound.

In other words, there exist real numbers m and M such that $m \leq f(x) \leq M$ for all x in the functions domine.

If there exist a positive integer M such that $|f(x)| \leq M$, for all x in the functions domine.

6.2 SOLVED EXAMPLES:

6.2.1 Example If f is continuous on [0,1] and if $\int_0^1 f(x)x^n dx = 0$ (n = 0,1,2,…),

prove that $f(x) = 0$ on [0,1].

(or)

If $\int_0^1 f(x)x^n dx = 0$ for all n in N. Show that $f = 0$ on [0,1].

Solution:

Assume that $\int_0^1 f(x)p(x)dx = 0$, for all polynomials $p(x)$ on [0,1].

By Stone – Weierstrass Theorem, there exists a sequence of polynomials p_n on C[0,1] which converges uniformly to f.

$\therefore \|p_n - f\|_\infty \to 0$

Since given integral is convergent, we have $\int_0^1 p_n f \to \int_0^1 f^2$.

But we have $\int_0^1 p_n f = 0$ for all n then $\int_0^1 f^2 = 0$ (Since $f \geq 0$)

$\therefore \int_0^1 f(x)dx = 0$ then $f = 0$ for all $x \in [0,1]$.

Hence $f = 0$.

6.2.2 Example: By Weierstrass approximation there is a sequence of polynomials $P_n{*}$ such that $\{P_n{*}\}$ converges uniformly on [-a,a] to $|x|$. Then $\lim_{n\to\infty} P_n{*}(0) = P{*}(0)$.

Solution:

If $P_n(x) = P_n{*}(x) - P_n{*}(0), \{P_n\}$ converges uniformly on [-a,a] to $|x|$ and $P_n(0) = 0$.

6.2.3 Example: Let $Q_n(x) = C_n(1-x^2)^n$. Find C_n if $\int_{-1}^{1} Q_n(x) = 1$.

Solution:

Let $Q_n(x) = C_n(1-x^2)^n$ then

$$Q_n(x) = C_n\left(1 - \binom{n}{1}x^2 + \binom{n}{2}x^4 - ... + (-1)^n x^{2n}\right).$$

Given 1 = $\int_{-1}^{1} Q_n(x)$ then

$$1 = C_n\left(\int_{-1}^{1} 1 - \binom{n}{1}\int_{-1}^{1} x^2 dx + \binom{n}{2}\int_{-1}^{1} x^4 dx - ... + (-1)^n \int_{-1}^{1} x^{2n} dx\right)$$

We know that $\int_{-1}^{1} x^{2k} dx = \left[\frac{x^{2k+1}}{2k+1}\right]_{-1}^{1} = \frac{2}{2k+1}$.

So $1 = C_n\left(2 - \binom{n}{1}\frac{2}{3} + \binom{n}{2}\frac{2}{5} - ... + (-1)^n \frac{2}{2n+1}\right)$

$$\Rightarrow 1 = 2C_n\left(1 - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - ... + \frac{(-1)^n}{2n+1}\right)$$

$$\Rightarrow C_n = \frac{1}{2}\left(1 - \frac{1}{3}\binom{n}{1} + \frac{1}{5}\binom{n}{2} - \ldots + \frac{(-1)^n}{2n+1}\right)^{-1}.$$

6.2.4 Example: Show that, if *f* is continuous on *R*, then there exist a sequence { P_n } of polynomials converging uniformly to *f* on each bounded subset of *R*.

Solution:

Given *f* is continuous on *R*.

Step 1

For each positive integer *n*, *f* is continuous on the interval [-*n*, *n*].

By the weierstrass Approximation theorem, there exist a polynomial $P_n(x)$ such that for all $x \in [-n, n]$, $|P_n(x) - f(x)| < \frac{1}{n}$.

This means P_n uniformly approximated *f* on [-*n*, *n*].

Step 2

Let *B* be any bounded sub set of *R*.

Then there exist an integer *N* such that $B \subseteq [-N, N]$.

For all $n \geq N$ and for all $x \in B$, we have $x \in [-n, n]$.

Therefore, $|P_n(x) - f(x)| < \frac{1}{n} \leq \frac{1}{N}$ for all $x \in B$ and $n \geq N$.

Given any $\varepsilon > 0$, choose *N* such that $\frac{1}{N} < \varepsilon$.

Then for all $n \geq N$ and for all $x \in B$, $|P_n(x) - f(x)| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

Thus, the sequence of polynomials { P_n } converges to *f* on *B*.

6.3 THEOREMS:

6.3.1 Theorem: State and Prove Stone – Weierstrass Theorem

If *f* is a continuous complex function on [a, b], there exist a sequence of polynomials {P_n} such that $\lim_{n \to \infty} P_n(x) = f(x)$ uniformly on [a, b]. If *f* is real, the P_n may be taken real.

Proof:

Let *f* be a complex valued continuous function on [a, b].

Without loss of generality, we may assume [a, b] = [0,1] and *f* (0) = *f* (1) = 0.

Since *f* continuous on [0,1] and [0,1] is compact .

$\Rightarrow f$ is uniformly continuous on [0,1].

$\Rightarrow$ For given $\in > 0 \; \exists \delta > 0 \ni |f(x) - f(y)| < \frac{\in}{2}$ when ever $|x-y| < \delta, \forall x, y \in [0,1]$ $\quad --\rightarrow$ (1)

Since f is continuous on [0,1].

$\Rightarrow f$ is bounded on [0,1]. (since every continuous function is bounded)

$\Rightarrow \exists$ a positive integer $M \ni |f(x)| \le M, \forall x \in [0,1]$. $\quad --\rightarrow$ (2)

Define $f(x) = 0, \forall x \notin [0,1]$

Clearly f is uniformly continuous on R.

Define $\phi_n(x) = C_n(1-x^2)^n$ where C_n is a constant $\ni \int_{-1}^{1} \phi_n(x)dx = 1$. $\quad --\rightarrow$ (3)

Consider $\int_{-1}^{1} (1-x^2)^n dx = 2\int_{0}^{1} (1-x^2)^n dx$

$$\ge 2\int_0^{1/\sqrt{n}} (1-x^2)^n dx \; (\because 1 \ge 1/\sqrt{n})$$

$$\ge 2\int_0^{1/\sqrt{n}} (1-nx^2)dx \; (\because (1-x)^n = 1 - nx + n(n-1)x/2 + ...)$$

$$= 2\left[x - n\frac{x^3}{3}\right]_0^{1/\sqrt{n}}$$

$$= 2\left[\frac{1}{\sqrt{n}} - \frac{n}{3}\left(\frac{1}{\sqrt{n}}\right)^3\right]$$

$$= 2\left[\frac{1}{\sqrt{n}} - \frac{n}{3}\frac{1}{n\sqrt{n}}\right]$$

$$= \frac{2}{\sqrt{n}}\left[1 - \frac{1}{3}\right]$$

$$= \frac{4}{3\sqrt{n}}$$

$$\ge \frac{1}{\sqrt{n}}$$

$$\therefore \int_{-1}^{1} (1-x^2)^n\, dx > \frac{1}{\sqrt{n}}$$

$$\Rightarrow \int_{-1}^{1} C_n (1-x^2)^n\, dx > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow \int_{-1}^{1} \phi_n(x) dx > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow 1 > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow \sqrt{n} > C_n$$

$$\Rightarrow C_n < \sqrt{n}$$

Now $\phi_n(x) = C_n(1-x^2)^n$

$$< \sqrt{n}(1-x^2)^n \,(\because C_n < \sqrt{n})$$

$$< \sqrt{n}(1-\delta^2)^n \,(\because \delta < x)$$

$$\therefore \phi_n(x) < \sqrt{n}(1-\delta^2)^n \to 0\, as\, n \to \infty.$$

$\Rightarrow \phi_n(x) \to 0\, as\, n \to \infty$ uniformly.

$\Rightarrow$ For given $\in > 0$ there exist positive integer $N \ni |\phi_n(x)| < \dfrac{\in}{8M} \forall n \geq N$ $\quad --\to (4)$

Define $P_n(x) = \int_{-1}^{1} f(x+t)\phi_n(t)dt$

$$= \int_{-1}^{-x} f(x+t)\phi_n(t)dt + \int_{-x}^{1-x} f(x+t)\phi_n(t)dt + \int_{1-x}^{1} f(x+t)\phi_n(t)dt \quad ---\to (5)$$

Consider $\int_{-1}^{-x} f(x+t)\phi_n(t)dt = \int_{x-1}^{0} f(v)\phi_n(v-x)dt = \int_{x-1}^{0} 0.\phi_n(v-x)dv = 0$ where $v = x + t$

$$\therefore \int_{-1}^{-x} f(x+t)\phi_n(t)dt = 0 \quad --\to (6)$$

Consider $\int_{-x}^{1-x} f(x+t)\phi_n(t)dt = \int_{0}^{1} f(v)\phi_n(v-x)dv \quad --\to (7)$

Consider $\int_{1-x}^{1} f(x+t)\phi_n(t)dt = \int_{1}^{x+1} 0\,\phi_n(v-x)dv = 0 \quad --\to (8)$

Substituting (6) (7) and (8) in (5),we get

$$p_n(x) = \int_{0}^{1} f(v)\,\phi_n(v-x)\,dv$$

$\therefore \{ p_n \}$ is sequence of polynomials on [0, 1].

To Prove $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on [0,1].

Let $x \in [0,1]$

Consider $\left|p_n(x)-f(x)\right| = \left|\int_{-1}^{1} f(x+t)\,\phi_n(t)dt - f(x)\right|$

$$\left|p_n(x)-f(x)\right| = \left|\int_{-1}^{1} f(x+t)\,\phi_n(t)dt - f(x)\int_{-1}^{1}\phi_n(t)dt\right| (\because By(3))$$

$$\left|p_n(x)-f(x)\right| = \left|\int_{-1}^{1}\left[f(x+t)-f(x)\right]\phi_n(t)dt\right|$$

$$\left|p_n(x)-f(x)\right| \le \int_{-1}^{1}\left|f(x+t)-f(x)\right|\left|\phi_n(t)\right|dt$$

$$= \int_{-1}^{-\delta}\left|f(x+t)-f(x)\right|\left|\phi_n(t)\right|dt + \int_{-\delta}^{\delta}\left|f(x+t)-f(x)\right|\left|\phi_n(t)\right|dt + \int_{\delta}^{1}\left|f(x+t)-f(x)\right|\left|\phi_n(t)\right|dt$$

$$\le \int_{-1}^{-\delta} 2M\left|\phi_n(t)\right|dt + \int_{-\delta}^{\delta}\left|f(x+t)-f(x)\right|\left|\phi_n(t)\right|dt + \int_{\delta}^{1} 2M\left|\phi_n(t)\right|dt \,(\because By(2))$$

$$< \int_{-1}^{-\delta} 2M\left|\phi_n(t)\right|dt + \int_{-\delta}^{\delta}\frac{\in}{2}\left|\phi_n(t)\right|dt + \int_{\delta}^{1} 2M\left|\phi_n(t)\right|dt \,(\because By(1))$$

$$< \int_{-1}^{-\delta} 2M\,\frac{\in}{8M}dt + \int_{-\delta}^{\delta}\frac{\in}{2}\left|\phi_n(t)\right|dt + \int_{\delta}^{1} 2M\,\frac{\in}{8M}dt \,(\because By(4))$$

$$= \frac{\in}{4}(1-\delta) + \frac{\in}{2}\int_{-\delta}^{\delta}\left|\phi_n(t)\right|dt + \frac{\in}{4}(1-\delta)$$

$$= \frac{\in}{2}(1-\delta) + \frac{\in}{2}\int_{-\delta}^{\delta}\left|\phi_n(t)\right|dt$$

$$< \frac{\in}{2}(1-\delta) + \frac{\in}{2}\int_{-1}^{1}\left|\phi_n(t)\right|dt$$

$$= \frac{\in}{2} + \frac{\in}{2}(1) \;(\because By(3))$$

$$= \in$$

$\therefore \left|p_n(x)-f(x)\right| < \in, \forall n \ge N, \forall x \in [0,1]$

$\Rightarrow \lim_{n\to\infty} P_n(x) = f(x)$ Uniformly on [0,1].

6.3.2 Corollary: For every interval [-a, a] there is a sequence of real polynomials { P_n } such that P_n (0) = 0 and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on [-a, a].

Proof:

Define f(x) = $|x|$ on [-a, a].

Clearly f is real valued function on [-a, a].

Then by Stone- weiestrass theorem there exist a sequence of real polynomials

$\{P_n{}^*\} \ni P_n{}^*(x) \to f(x)$ uniformly on [-a, a].

$\Rightarrow P_n{}^* \to |x|$ Uniformly on [-a, a].

If x = 0 then $P_n{}^*(0) \to 0$ uniformly on [-a, a].

Define a polynomial $P_n(x) = P_n{}^*(x) - P_n{}^*(0)$

Now $P_n(0) = P_n{}^*(0) - P_n{}^*(0) = 0$.

$\therefore P_n(0) = 0$

Consider $|P_n(x) - |x|| = |P_n{}^*(x) - P_n{}^*(0) - |x|| \to ||x| - 0 - |x|| \to 0$ uniformly on [-a, a].

Hence $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on [-a, a].

Therefore there exist a sequence of real polynomials { P_n } such that P_n (0) = 0 and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on [-a, a].

6.4 SUMMARY:

The Stone-Weierstrass theorem is generalization of Weierstrass approximation.
The Stone-Weierstrass theorem, a cornerstone of mathematical analysis, stated that any continuous function on a compact set can be approximated to ant desired degree of accuracy by a polynomial function.

6.5 TECHNICAL TERMS:

- Sequence
- Complex valued function
- Bounded function
- Continuous function
- Compact set

- Uniformly convergent
- Stone-Weierstrass theorem

6.6 SELF- ASSESSMENT QUESTIONS:

1. If f is continuous on [0,1] and if $\int_0^1 f(x)x^n dx = 0$ (n = 0,1,2,…), prove that $f(x) = 0$ on [0,1].

2. Find $S_n(x) = \int_0^2 tQ_n(t-x)dx$ for n = 1,2,3,4.

   The sequence S_n $(n \geq 1)$ converges to $|x|$ on [-2,2].

   The sequence $R_n(x) = \int_0^1 tQ_n(t-x)dx$, for n = 1,2,3,4 converges to $|x|$ on [-1,1].

   We know these facts from Weiestrass approximation. Is S_n restricted to [-1,1] equal to R_n ?

3. Show that there does not exist a sequence of polynomials converging uniformly on R to f, where $f(x) = e^x$.

4. Show that there does not exist a sequence of polynomials converging uniformly on R to f, where $f(x) = \sin x$.

6.7 SUGGESTED READINGS:

1. Principles of Mathematical Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M. Apostal, Narosa Publishing House, 2nd Edition, 1985.

- Dr. N.S.L.V. Narasimharao.

LESSON- 7
ALGEBRA OF FUNCTIONS

OBJECTIVES:

After studying the lesson, you should able to illustrate algebra of complex or real function uniform closure of complex or real function and the stone generalization of the weierstrass Theorem.

STRUCTURE:

7.1 INTRODUCTION:

In this Lesson, we define and study the Algebra of functions, uniformly closed Algebra and some examples. There are many definitions which leads some theorems

7.2 DEFINITIONS:

7.2.1 Definition:

A family A of complex functions defined on a set E is said to be an algebra if

i) $f + g \in A,$

ii) $fg \in A, \; and$

iii) $cf \in A \; forall \; f; \; g \in A \; and \; forall$

Complex constants C (that if A is closed under addition, multiplication and scalar multiplication)

7.2.2 Definition (Uniformly Closed):

A family A of complex functions defined on a set E is said to be uniformly closed, if A has the property $f \in A$, whenever $f_n \in A \; (n = 1, 2, 3 ..)$ and $f_n \to f$ uniformly on E.

7.2.3 Example: The set of all polynomials is an algebra.

7.2.4 Definition (Uniformly Closure)

Let A be an algebra of all complex functions defined on a set E. The set B of all function which are limits of uniformly convergent sequence of members of A , is called The uniform closure of A.

7.2.5 Example:

Let [a,b] a closed interval in R. then the set A of all polynomials defined on [a,b] is an algebra. By the stone-Weierstrass theorem, the set of continuous functions on [a,b] is the uniform closure of A

7.3 ALGEBRA OF FUNCTIONS:

7.3.1 Theorem

Let B be the uniform closure of an algebra A of bounded functions, then B is an uniformly closed algebra.

Proof:

Let A be an algebra of bounded functions defined on a a set E, and let B be the uniform closure of A. First, we show that B is an algebra

Let for all $f, g \in B, \ and \ \ c \in C$

Since B is the uniform closure of A, there exist sequences $\{f_n\}$ and $\{g_n\}$ in A such that $f_n \to f$ and $g_n \to g$ uniformly on E.

i) let $\in > 0$ be given

Since $f_n \to f$ uniformly on E, $\exists$ a +ve integer

$$N_1 \ni |f_n(x) - f(x)| < \frac{\in}{2} \ \ \forall \, n \geq N_1 \ and \ \forall \, x \in E \to (1)$$

Since $g_n \to g$ uniformly on E, $\exists$ a +ve integer

$$N_2 \ni |g_n(x) - g(x)| < \frac{\in}{2} \ \ \forall \, n \geq N_2 \ and \ \forall \, x \in E \to (2)$$

As f_n and g_n are bounded, $f_n + g_n$ is also bounded and so $f_n + g_n \in A$ for n=1,2,3….

Clearly $\therefore \{f_n + g_n\}$ is a sequence in A.

Write $N = \max\{N_1, N_2\} \to (3)$

Now for all $n \geq N$ and for all $x \in E$, consider

$$\left|(f_n+g_n)(x)-(f-g)(x)\right| = \left|f_n(x)+g_n(x)-f(x)-g(x)\right|$$

$$\leq \left|f_n(x)-f(x)\right|+\left|g_n(x)-g(x)\right|$$

$$< \frac{\in}{2}+\frac{\in}{2} \quad = \in$$

(from (1),(2) and (3))

This shows that $f_n + g_n \to f + g$ uniformly on E

$\therefore f+g \in B$

ii) since f and g are bounded, $\exists$ real numbers M_1 and M_2 such that

$|f(x)| \leq M_1$ and $|g(x)| \leq M_2 \ \forall \ x \in E \quad \to (4)$

Now we show that $f_n g_n \to fg$ uniformly on E

Since $f_n \to f$ uniformly on E, $\exists$ a +ve integer N, $\ni |f_n(x) - f(x)| < \frac{\in}{2(M_1+M_2+1)}$

For all $n \geq N_1$ and for all $x \in E \quad \to (5)$

Consider $|f_n(x)| = |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)|$

$$< \frac{\in}{2(M_1+M_2+1)} + M_1, \ \forall \ x \in E \ \ and \ \ \forall \ n \geq N_1$$

(from (4) and (5))

Since $g_n \to g$ uniformly on E, $\exists$ a +ve integer N_2 such that

$$|g_n(x) - g(x)| < \frac{\in}{2\left(\frac{\in}{(M_1+M_2+1)} + M_1\right)}, \ \forall \ n \geq N_2 \ \ and \ \ \forall \ x \in E$$

Write $N = \max \{N_1, N_2\} \to (6)$

For all $n \geq N$ and for all $x \in E$, consider

$$\left|(f_n g_n)(x) - (fg)(x)\right| = \left|f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)\right|$$

$$\leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)|$$

$$< \left(\frac{\in}{2(M_1+M_2+1)} + M_1\right)\left(\frac{\in}{2\left(\frac{\in}{M_1+M_2+1} + M_1\right)} + M_1\right) + M_2 \frac{\in}{2(M_1+M_2+1)}$$

$$< \frac{\in}{2}+\frac{\in}{2} = \in$$

(from (4),(5) and (6))

$\therefore \exists$ a +ve integer N such that $\left|(f_n g_n)(x) - (fg)(x)\right| < \in \quad \forall \ n \geq N \ \ and \ \ \forall \ x \in E$

Hence, $f_n g_n \to fg$ uniformly on E so, $fg \in B$

iii) let C be any constant and $f \in B$

since $f \in B$, $\exists$ a sequence $\{f_n\}$ in A $\ni f_n \to f$ uniformly on E

since A is an algebra, $cf_n \in A \;\forall\; n$

if $C = 0$, then clearly $cf_n \to cf$ uniformly on E

if $C \neq 0$, then $|C| > 0$.

Since $f_n \to f$ uniformly on E, $\exists$ a +ve integer $N \ni |f_n(x) - f(x)| < \frac{\in}{|c|}$

$\forall \; n \geq N \;\; and \;\; \forall \; x \in E \to (7)$

so, for $n \geq N$ and for $x \in E$,

consider $|(cf_n)(x) - (cf)(x)| = |c||f_n(x) - f(x)| < |c|\frac{\in}{|c|} = \in$ (from(7))

$\Rightarrow |(cf_n)(x) - (cf)(x)| < \in \;\forall\; n \geq N \;\; and \;\; \forall \; x \in E$

$\therefore cf_n \to cf$ uniformly on E. so, $cf \in B$

Hence B is an algebra.

Finally, we show that B is uniformly closed

Let $\{f_n\}$ be a sequence in B $\ni f_n \to f$ uniformly on E

Since each $f_n \in B$, we have that each f_n is bounded for $n \geq 1$,

And so each $f_n \in A$ for $n \geq 1$.

Now $\{f_n\}$ is a sequence in A $\ni f_n \to f$ uniformly on E

Since B is the uniform closure of A, we have $f \in B$

Hence, B is uniformly closure of A, we have $f \in B$.

7.3.2 Definition:

Let A be a family of functions defined on a set E. then A is said to separate point on E if to every pair of distinct point $x_1, x_2 \in E$, there corresponds a function $f \in A$ such that $f(x_1) \neq f(x_2)$.

7.3.3 Definition (Vanishes at no point of E)

If to each $x \in E$ there corresponds a function $g \in A$ such that $g(x) \neq 0$, we say that A vanishes at no point of E.

7.3.4 Note: the algebra of all polynomials in one variable clearly has these properties on R

7.3.5 Example:

The set of all even polynomials define on [-1,1] is an algebra which doesn't separate points Since $f(-x) = f(x)$ for every even polynomial f but $-x \neq x \;\forall\; 0 \neq x \in [-1,1]$

7.3.6 Theorem

Suppose A is an algebra of function on a set E, A separates points on E, and A vanishes at no point of E. suppose x_1, x are distinct point of E , and c_1, c_2 are constants (real if A is real algebra) .then A contains a function f such that $f(x_1) = c_1$, $f(x_2) = c_2$.

Proof:

let A an algebra of functions defined on a set E

Since A separates points on E, there is a function g in $A \ni g(x_1) = g(x_2)$

Since A vanishes at no point of E, there exist h, $K \in A \ni h(x_1) \neq 0 \text{ and } K(x_2) \neq 0$

Put $u = gk - g(x_1)k$, $\vartheta = gh - g(x_2)h$

Then $u \in A \text{ and } \vartheta \in A$

Now $u(x_1) = \left(gk - g(x_1)k\right)(x_1) = g(x_1)k(x_1) - g(x_1)k(x_1) = 0$ and

$u(x_2) = g(x_2)k(x_2) - g(x_1)k(x_2) \neq 0$

$||^{ly}$ $\vartheta(x_1) \neq 0$ and $\vartheta(x_2) = 0$

Put $f = \dfrac{c_1\vartheta}{\vartheta(x_1)} + \dfrac{c_2 u}{u(x_2)}$.

Then $f \in A$ and $f(x_1) = \dfrac{c_1\vartheta(x_1)}{\vartheta(x_1)} + \dfrac{c_2 u(x_1)}{u(x_2)} = c_1 + 0 = c_1$,

$$f(x_2) = \frac{c_1\vartheta(x_2)}{\vartheta(x_1)} + \frac{c_2 u(x_2)}{u(x_2)} = 0 + c_2 = c_2$$

Thus, $\exists\, f \in A \ni f(x_1) = c_1 \text{ and } f(x_2) = c_2$

The following theorem is the stones generalization of the weierstrass theorem:

7.3.7 Theorem:

Let A be an algebra of real continuous functions on a compact set k. If A separates points on K and if A vanishes at no point of K, then the uniform closed B of A consists of all real continuous function on K.

Proof:

Let A is an algebra of real; continuous functions on a compact set K

Suppose that A separates point on K and A vanishes at no point of K.

Let B be the uniform closed of A

Since A is an algebra of continuous function on K and K is compact, by a know theorem (If f is a continuous function of a compact metric space X into R^K, then f(x) is closed and bounded and hence f is bounded) every member of A is a bounded function on K.

Since B is the uniform closure of A , by the theorem (Let B be the uniform closure of an algebra bounded functions. Then B is a uniformly closed algebra),

B is an uniformly closed algebra.

We shall divide the proof into four steps

Step 1:

If $f \in B$, then $|f| \in B$

Let $f \in B$

Put $a = \sup_{x \in k} |f(x)|$. Consider the closed interval [-a,a] in R

By a known corollary (For every interval [-a,a] there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that $\lim_{n\to\infty} P_n(x) = |x|$ uniformly on [-a,a]) for [-a,a], there exist a sequence of real polynomial P_n define by $P_n(y) = \sum_{i=1}^{n} c_i y^i$ such that $P_n(0) = 0$ and

$\lim_{n\to\infty} P_n(y) = |y|$ uniformly on [-a,a] -→(1)

Take $\in > 0$

Then by (1), $\exists$ a +ve integer N such that $\left|P_n(y) - |y|\right| < \in \ \forall\ n \geq N \ \forall\ y \in [-a,a]$

$\Rightarrow \left|P_N(y) - |y|\right| < \in \ \forall -a \leq y \leq a$

$\Rightarrow \left|\sum_{i=1}^{N} c_i y^i - |y|\right| < \in \ \forall\ -a \leq y \leq a \rightarrow (2)$

Put $g = \sum_{i=1}^{N} c_i f^i$

Since B is an algebra, then $g \in B$

Since $|f(x)| \leq a, \forall\ x \in K,$ by (2) ,

we have $\left|g(x) - (|f(x)|)(x)\right| = \left|\sum_{i=1}^{N} c_i\,(f(x))^i - |f(x)|\right| < \in \ \forall\ x \in K$

$$\Rightarrow \left|g(x) - |f|(x)\right| < \in \ \forall\ x \in K$$

This shows that the constant sequence $\{g\}$ converges to $|f|$ uniformly on K

So, $|f| \in B$.

Step 2:

In this step, we prove that, if $f \in B$ and $g \in B$, then $\max(f,g)$ and $\min(f,g)$ are in B

Let $f, g \in B$.

By $\max(f,g)$, we mean the function h define by

$$h(x)=\begin{cases} f(x) & if \quad f(x)\ge g(x), \\ g(x) & if \quad f(x)\le g(x); \end{cases} \quad \text{and}$$

$\min(f,g)$, we mean the function K defined by

$$K(x)=\begin{cases} f(x) & if \quad f(x)\le g(x), \\ g(x) & if \quad f(x)\ge g(x); \end{cases}$$

Then $\max(f,g)=\dfrac{f+g}{2}+\dfrac{|f-g|}{2}$ and $\min(f,g)=\dfrac{f+g}{2}-\dfrac{|f-g|}{2}$

Since B is an algebra and $f, g \in B$, $f-g \in B$.

So, by step1, $|f-g| \in B$

$\therefore \max(f,g), \min(f,g) \in B$

By induction, we can prove that if $f_1, f_2, ...f_n \in B$,

then $\max(f_1, f_2, ...f_n)$ and $\min(f_1, f_2, ...f_n) \in B$.

Step 3:

Prove that given a real function f continuous on K, a point $x \in K$, and $\in > 0$, there exist a function $g_x \in B$ such that $g_x(x) = f(x)$ and $g_x(t) > f(t) - \in \ \forall\, t \in K$

Let f be a real continuous function on K, $x \in K$ and $\in > 0$. Clearly, $A \subseteq B$

Claim: B separates point on K and B vanishes at no point of K

Let $x_1, x_2 \in K \ni x_1 \neq x_2$

Since A separates point on K, then $\exists\ g \in A \ni g(x_1) \neq g(x_2)$ as $A \subseteq B$, $g \in B$

Since B separates point on K

Let $y \in K$

Since A vanishes at no point on K, $\exists\ h \in A \ni h(y) \neq 0$

As $A \subseteq B$, $h \in B$

$\therefore$ B vanishes at no point of K

Hence B is an algebra of function on K, such that B separates point on K and B vanishes at no point of K.

So, by a known theorem (7.26) for every $y \in K$, we can find a function $h_y \in B \ni h_y(x) = f(x) \text{ and } h_y(y) = f(y)$

$\Rightarrow \left(f - h_y\right)(y) = 0 < \in$

Since $h_y - f$ is continuous at y , $\exists$ an open set J_y of

$y \ni \left|\left(h_y - f\right)(t) - \left(h_y - f\right)(y)\right| < \in \ \forall\ t \in J_y$

$\Rightarrow \left|h_y(t) - f(t)\right| < \in \ \forall\ t \in J_y$

$\Rightarrow - \in < h_y(t) - f(t) < \in \ \ \forall\ t \in J_y$

$\Rightarrow h_y(t) - f(t) < \in \ \ \forall\ t \in J_y \rightarrow (3)$

Now the family $\left\{J_y / y \in K\right\}$ of open sets is an open cover for K

Since K is compact, $\exists\ y_1, y_2, y_3, ..., y_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} J_{y_j}$

Put $g_x = \max\left\{h_{y_1}, h_{y_2}, h_{y_3}, ..., h_{y_n}\right\}$

Since each $h_{y_1} \in B$ for i=1,2,3,….n , by step 2, $g_x \in B$

Consider $g_x(x) = \max\left\{h_{y_1}(x), h_{y_2}(x), ..., h_{y_n}(x)\right\}$

$= \max\left\{f(x), f(x), ..., f(x)\right\}$

$= f(x)$

Consider $g_x(t) = \max\left\{h_{y_1}(t), h_{y_2}(t), ..., h_{y_n}(t)\right\} > f(t) - \in$

$\therefore g_x(t) > f(t) - \in \forall\ t \in K$

Thus, for every real continuous function f on K and a point $x \in K$

and $\in > 0$, $\exists$ a function $g_x \in B \ \ni g_x(x) = f(x)$ and $g_x(t) > f(t) - \in, \forall\ t \in K$

Step 4:

In this step, we prove that for a given real continuous function f on K and $\in > 0$, $\exists$ a function $h \in B \ni \left|h(x) - f(x)\right| < \in (x \in K)$

Let f be a real continuous function on K. let $\in > 0$

For each $x \in K$, consider the function g_x, constructed as in step(3) by the continuity of g_x, there exist open sets V_x containing x,

Such that $|g_x(t) - f(t)| < \in (t \in V_x)$

$\Rightarrow - \in < g_x(t) - f(t) < \in$ for all $t \in V_x$

$\Rightarrow g_x(t) < f(t) + \in \ \forall \ t \in V_x \rightarrow (4)$

Now the family $\{V_x / x \in K\}$ of open sets forms an open cover for K

Since K is compact, $\exists \ x_1, x_2, x_3, ..., x_n$ in $K \ni K = \bigcup_{i=1}^{n} V_{x_i}$

Take $h = \min\{g_{x_1}, g_{x_2}, ..., g_{x_n}\}$

Since each $g_{x_i} \in B$ for i=1 to n , by step(2), $h \in B$

For any $t \in K$, $h(t) = \min\{g_{x_1}(t), g_{x_2}(t), ..., g_{x_n}(t)\} > f(t) - \in$

$\Rightarrow h(t) > f(t) - \in \ \forall \ t \in K \rightarrow (5)$ (by step(3))

Let $t \in K = \bigcup_{i=1}^{n} V_{x_i}$

$\Rightarrow t \in V_{x_i}$ for some $1 \le i \le n$

$\Rightarrow g_{x_i}(t) < f(t) + \in$ (by (4))

$h(t) = \min\{g_{x_1}(t), g_{x_2}(t), ..., g_{x_n}(t)\} < f(t) + \in \forall \ t \in K \rightarrow (6)$

Form (5) & (6), we have $|h(t) - f(t)| < \in \forall \ t \in K$

Since B is uniformly closed algebra, we have $f \in B$

Hence, the uniform closure B of A consists of all real continuous function on K

7.3. 8 Definition:

An algebra A of complex function defined on a set E is said to be self-adjoint, if for every $f \in A$, its complex conjugate $\overline{f} \in A$.

7.3.9 Theorem:

Suppose A is a self-adjoint algebra of complex continuous function on a compact set K, A separates points on K, and A vanishes at no point of K. then the uniform closure B of A consists of all complex continuous function on K. in other words, A is dense in ξ (K).

Proof: suppose A is a self-adjoint algebra of complex continuous functions on a compact set K.

Also given that A separates point on K

Let A_R be the set of all real continuous function on K which belong to A

i.e $A_R = \{ f \in A / f \, is \, a \, real \, continuous \, function \, on \, K \}$

Let $f \in A$

Then $f = u + i\vartheta$ where u and ϑ are real continuous function on K

Since A is self-adjoint, then $u - i\vartheta = \overline{f} \in A$

So, $u = \dfrac{f + \overline{f}}{2} \in A$ and $\vartheta = \dfrac{f - \overline{f}}{2i} \in A$

Claim: A_R is an algebra

Let $f, g \in A_R \Rightarrow f, g \in A$

Since A is an algebra, by definition $f + g, fg \; and \; cf$ are in A for any real constant C

It is clear that $f + g, fg \; and \; cf$ are real continuous function on K

So, $f + g, fg \; and \; cf$ are in A_R for any real constant C

$\therefore A_R$ is an algebra

Claim: A_R separates point on K

Let $x_1, x_2 \in K \ni x_1 \neq x_2$

Since A separates point on K, then $\exists f \in A \ni f(x_1) \neq f(x_2)$

$\Rightarrow$ either $(\text{Re}.f)(x_1) \neq (\text{Re}.f)(x_2)$ or $(\text{Im}.f)(x_1) \neq (\text{Im}.f)(x_2)$ $\rightarrow$(1)

Where Re.f and Im.f are real and imaginary parts of f respectively

Since Re.f and Im.f are functions in A_R, by (1) A_R separates point on K

Claim: A_R vanishes at no point of K

Since A vanishes at no point of K, for $x \in K, \exists f \in A \ni f(x) \neq 0$

$\Rightarrow$ either $(\text{Re}.f)(x) \neq 0$ or $(\text{Im}.f)(x) \neq 0$

As Ref, Imf $f \in A_R$, A_R vanishes at no point of K

$\therefore A_R$ satisfies the hypothesis of stones generalization of weierstrass theorem

It follows that every real continuous function on K is in the uniform closure of A_R and hence in B

If f is a complex continuous function on K and $f = u + i\vartheta$ where u and ϑ are real and imaginary parts of f respectively, then u and ϑreal continuous function on K,
i.e., $u, \vartheta \in B$ and hence $f \in B$

This completes the proof.

7.4 SUMMARY:

This lesson is designed to introduce learners to the fundamental concept the Algebra of functions, exploring their properties and applying them to real-world contexts. This lesson provides a solid foundation for learners to develop their understanding. Key takeaways of their lesson are definitions and theorems, applications of the Algebra of functions in mathematical and real-world problems and examples.

7.5 TECHNICAL TERMS:

- Algebra of functions
- Uniformly closure
- Self-adjoint

7.6 SELF-ASSESSMENT QUESTIONS:

1. Let K be the unit in the complex plane (i.e, the set of all z with |z| =1) and $f\left(e^{i\theta}\right) = \sum_{n=0}^{N} C_n e^{in\theta}$ (θ $real$), Then A separates points on K and A vanishes at no points of K, but never the less there are continuous on K which are not in the uniform closure of **Hint:** For every $f \in A$, $\int_0^{2\pi} f\left(e^{i\theta}\right) e^{i\theta} d\theta = 0$, and this is also true for every f in the closure of A.

2. If f is continuous on [0,1] and if $\int_0^1 f(x)\, x^n dx = 0 \ (n = 0,1,2,3,...)$, prove that $f(x) = 0$ on [0,1]

3. If $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2} (0 \le x \le 1,\ n = 1,2,...)$, then show that

   i) $\{f_n\}$ is uniformly bounded on [0,1]

   ii) No sub sequence of $\{f_n\}$ is converge uniformly on [0,1],

   iii) The sequence $\{f_n\}$ is equi continuous on [0,1]

7.7 SUGGESTED READINGS:

1. Principles of mathematical analysis by Walter Rudin , 3rd Edition

2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985

- **Dr. L. Krishna**

LESSON- 8
POWER SERIES

OBJECTIVES:

The objective of the lesson is to understand the concepts of the power series. We shall derive some properties of functions which are represented by power series

STRUCTURE:

8.1 Introduction

8.2 Definition

8.3 Power Series

8.4 Summary

8.5 Technical Terms

8.6 Self-Assessment Questions

8.7 Suggested Readings

8.1 INTRODUCTION:

In this lesson we shall derive some properties of functions which represented by power series, that is functions of the form $f(x)=\sum_{n=0}^{\infty} C_n x^n$, which convergence uniformly on some interval and different definitions such as converges uniformly and it's derivatives

8.2 DEFINITION:

A series of the form $\sum_{n=0}^{\infty} a_n x^n$ where the 'a_n' are independent of 'x' is called a power series in x

8.2.1 Result

1) let $f(x)=\sum_{n=0}^{\infty} a_n x^n \rightarrow (1)$

If the series given in (1) converges for all x in (-R,R) , for some R>0 (R may be $+\infty$), we say that f is expanded in a power series about the point x=0

2) let $f(x)=\sum_{n=0}^{\infty} a_n (x-a)^n \rightarrow (2)$

If the series given in (2) converges for all x with $|x-a|<R$, foe some R>0, we say that f is expanded in a power series about the point x=a

8.2.2 Note:

The number R associated with the power series given in (1) is called the radius of convergence of the series and it is defined as $R = \frac{1}{\alpha}$ where $\alpha = \limsup_{n\to\infty} \sqrt[n]{a_n}$. The interval (-R,R) is called the interval of convergence of the series.

8.3 POWER SERIES:

8.3.1 Theorem

Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n \ \left(|x| < R\right)$, then $\sum_{n=0}^{\infty} c_n x^n$ converges uniformly on $[-R+\in, R-\in]$, no matter which $\in > 0$ is chosen. The function f is continuous and differentiable in (-R,R), and $f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \ \left(|x| < R\right)$.

Proof:

Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$

Define $f(x) = \sum_{n=0}^{\infty} c_n x^n \ \left(|x| < R\right)$.

i)let $\in > 0$ consider the interval $[-R+\in, R-\in]$

let $x \in [-R+\in, R-\in]$ then $|x| < R-\in$

so, for $n \geq 0$ $|c_n x^n| = |c_n||x^n| \leq |c_n|(R-\in)^n \rightarrow (1)$

since every power series converges absolutely in the interior of its interval of convergence, $\sum_{n=0}^{\infty} c_n (R-\in)^n$ converges absolutely

since $|c_n x^n| \leq |c_n (R-\in)^n| \ \forall \ n \geq 0$ and $\sum_{n=0}^{\infty} |c_n (R-\in)|^n$ is convergent by a known theorem(Suppose f_n is a sequence of functions defined on E, and suppose $|f_n(x)| \leq m_n \ (x \in E, \ n = 1,2,3,....)$. Then $\sum f_n$ converges uniformly on E if $\sum m_n$ converges)

$\sum_{n=0}^{\infty} c_n x^n$ converges on $[-R+\in, R-\in]$.

ii) since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, we have $\limsup_{n\to\infty} \sqrt[n]{n|c_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$

$\therefore$ The two series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} n c_n x^{n-1}$ have the same interval of converges

Since $\sum_{n=0}^{\infty} nc_n x^{n-1}$ is a power series which converges for $|x| < R$, by the first part, $\sum_{n=0}^{\infty} nc_n x^{n-1}$ converges uniformly on $[-R+\in, R-\in]$ for all $\in>0$ for $n \geq 0$, write $f_n(x) = c_n x^n$ then $f'(x) = nc_n x^{n-1}$

Since $f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $[-R+\in, R-\in]$ for all $\in>0$,

$\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[-R+\in, R-\in]$ and , we have $f'(x) = \sum_{n=0}^{\infty} f_n'(x)$

$= \sum_{n=0}^{\infty} nc_n x^{n-1}$

But given any x such that $|x| < R$, we can find an $\in>0$ such that $|x| < R-\in$ which shows that

$f'(x) = \sum_{n=0}^{\infty} nc_n x^{n-1} \; for \; (|x| < R)$

Since f is differentiable, by a known result, f is continuous.

8.3.2 Corollary:

Suppose series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and define $f(x) = \sum_{n=0}^{\infty} c_n x^n \; (|x| < R)$. Then f has derivatives of all orders in (-R,R), which are given by

$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)......(n-k+1)c_n x^{n-k}$

In particular, $f^{(k)}(0) = k!\,c_K$ (k=0,1,2,…..)

(here $f^{(0)}$ means f, and $f^{(k)}$ is the K[th] derivatives of f, for K=1,2,3….)

Proof:

Suppose the series $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < R$ and defined $f(x) = \sum_{n=0}^{\infty} c_n x^n$ n=0

Then by the above theorem(8.21), f is differentiable in (-R,R) and $f'(x) = C_1 + 2C_2 x + 3C_3 x^2 + ... \; for \quad |x| < R$

Again by applying the same the and continuing the process, we get f is differentiable in (-R,R) and $f^{(k)}(x) = k!\,c_K + ... + n(n-1)(n-2)....(n-k+1)c_n x^{n-k} + ... + ..., \; for \quad |x| < R$

Putting x=0, $f^{(k)}(0) = k!\,c_K$, for K=0,1,2,3….

8.3.3 Theorem:

(Abels Theorem): suppose $\sum_{n=0}^{\infty} C_n$ converges, put $f(x)=\sum_{n=0}^{\infty} C_n x^n \ (-1<x<1)$ then $\lim_{x\to 1} f(x)=\sum_{n=0}^{\infty} C_n$.

Proof:

Suppose $\sum_{n=0}^{\infty} C_n$ converges

Let $\sum_{n=0}^{\infty} C_n = s$

For each n, write $s_n = C_0 + C_1 + ... + C_n$, and $s_{-1} = 0$

Then $\lim_{n\to\infty} s_n = s \rightarrow (1)$

Set $\in > 0$ be given from(1), $\exists$ a +ve integer $N \ni |s_n - s| < \frac{\in}{2} \ \forall \ n \geq N \rightarrow (2)$

Now $s_n = C_0 + C_1 + ... + C_{n-1} + C_n = s_{n-1} + C_n \ \forall \ n \geq 0$

$\Rightarrow C_n = s_n - s_{n-1} \ \forall \ n \geq 0$

$\therefore \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} (s_n - s_{n-1}) x^n$

For any m, $\sum_{n=0}^{m} (s_n - s_{n-1}) x^n = (s_0 - s_{-1}) + (s_1 - s_0)x + (s_2 - s_1)x^2 + ... + (s_m - s_{m-1})x^m$

$$= s_0 - s_{-1} + s_1 x - s_0 x + s_2 x^2 - s_1 x^2 + ... + s_m x^m - s_{m-1} x^m$$

$$= (1-x)\left[s_0 + s_1 x + s_2 x^2 + ... + s_{m-1} x^{m-1} \right] + s_m x^m$$

$$= (1-x)\sum_{n=0}^{m-1} s_n x^n + s_m x^m$$

$$\therefore \sum_{n=0}^{m} (s_n - s_{n-1}) x^n = (1-x)\sum_{n=0}^{m-1} s_n x^n + s_m x^m \rightarrow (3)$$

For $|x| < 1, \lim_{m\to\infty} s_m x^m = \lim_{m\to\infty} s_m . \lim_{m\to\infty} x^m = s.0 = 0 \rightarrow (4)$

Put $f(x) = \sum_{n=0}^{\infty} C_n x^n \ (-1 < x < 1)$

for $|x| < 1, \ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$\Rightarrow (1-x)\sum_{n=0}^{\infty} x^n = 1$, for $|x| < 1 \rightarrow (5)$

Fix $|x| < 1$

Consider $f(x) = \sum_{n=0}^{\infty} C_n x^n = \lim_{m\to\infty} \sum_{n=0}^{\infty} C_n x^n$

$$= \lim_{m\to\infty} \sum_{n=0}^{m} (s_n - s_{n-1}) x^n$$

$$= \lim_{m\to\infty} (1-x) \sum_{n=0}^{m-1} s_n x^n + \lim_{m\to\infty} s_m x^m$$

$$= \lim_{m\to\infty} (1-x) \sum_{n=0}^{m-1} s_n x^n$$

$$\therefore f(x) = \lim_{m\to\infty} (1-x) \sum_{n=0}^{m-1} s_n x^n = \sum_{n=0}^{\alpha} s_n x^n (1-x)$$

Hence $f(x) - s = f(x) - s.1$

$$= f(x) - s(1-x)\sum_{n=0}^{\infty} x^n \quad (by\ (2)\ )$$

$$= (1-x)\sum_{n=0}^{\infty} s_n x^n (1-x) - s(1-x)\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} x^n (1-x)(s_n - s)$$

Now , $|f(x) - s| = \left| \sum_{n=0}^{\infty} x^n (1-x)(s_n - s) \right|$

$$\leq \sum_{n=0}^{\infty} |x^n| |1-x| |s_n - s|$$

$$\therefore |f(x) - s| \leq \sum_{n=0}^{n-1} |x^n| |1-x| |s_n - s| + \leq \sum_{n=N}^{\infty} |x^n| |1-x| |s_n - s|$$

$$< |1-x| \sum_{n=0}^{n-1} |s_n - s| |x^n| + |1-x| \frac{\in}{2} \sum_{n=N}^{\infty} |x^n| \quad (by\ (2)\ ) \rightarrow (6)$$

Since $|1-x| \sum_{n=0}^{N-1} |s_n - s| |x^n|$ is continuous at x=1, then $\exists$ a $\delta_1 > 0$

Such that $|1-x| \sum_{n=0}^{N-1} |s_n - s| |x^n| < \frac{\in}{2}$ for all x with $1 - \delta_1 < x < 1 + \delta_1 \rightarrow (7)$

We have $\left|1-x\right|\sum_{n=N}^{\infty}\left|x^n\right| = \left|1-x\right|\left|x^N(1+x+x^2+...)\right|$

$$=\left|1-x\right|\left|x^N\right|\frac{1}{\left|1-x\right|} = x^N < 1 \text{ for all x with } 1-\delta_1 < x < 1+\delta_1 \rightarrow (8)$$

So, from(6),(7),(8), $\left|f(x)-s\right| < \frac{\in}{2}+\frac{\in}{2} = \in$ for all $1-\delta_1 < x < 1+\delta_1$

Thus, $\exists \delta_1 > 0 \ni \left|f(x)-s\right| > \in$ when ever $1-\delta_1 < x < 1+\delta_1$

$\Rightarrow \lim_{x\to 1} f(x) = s$

i.e., $\lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} C_n$

8.3.4 Corollary:

Suppose the series $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n, \sum_{n=0}^{\infty} c_n$ converge to A,B,C respectively for each $n \geq 0$, put $C_n + a_0 b_n + a_1 b_{n-1} + ... + a_n b_0$ then C=AB

Proof:

let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $g(x) = \sum_{n=0}^{\infty} b_n x^n$, $h(x) = \sum_{n=0}^{\infty} c_n x^n$ for $0 \leq x \leq 1$

for $0 \leq x \leq 1$, these series converge absolutely and hence may be multiplied

so, we have $f(x)g(x) = \sum_{n=0}^{\infty} a_n x^n . \sum_{n=0}^{\infty} b_n x^n$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + ...$$

$$= c_0 + c_1 x + c_2 x^2 + ... \quad \text{where } C_n = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0, \forall\, n \geq 0$$

$$= \sum_{n=0}^{\infty} C_n x^n = h(x) \qquad (0 \leq x \leq 1) \rightarrow (1)$$

By the theorem(8.23), $f(x) \to A, g(x) \to B, h(x) \to C$ as $x \to 1$

From(1), $\lim_{x\to 1} f(x)g(x) = \lim_{x\to 1} h(x)$

$\Rightarrow \lim_{x\to 1} f(x). \lim_{x\to 1} g(x) = \lim_{x\to 1} h(x)$

$\Rightarrow AB = C$

8.3.5 Theorem

Given a double sequence $\{a_{ij}\}$, $i=1,2,3,....$, $j=1,2,3,.....$,

Suppose that $\sum_{j=1}^{\infty}|a_{ij}|=b_i$ $(i=1,2,3,....)$ and $\sum b_i$ converges. Then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$

Proof:

Given a double sequence $\{a_{ij}\}$, $i=1,2,3,....$ and $j=1,2,3,.....$,

Suppose that $\sum_{j=1}^{\infty}|a_{ij}|=b_i$ for $i=1,2,3,....$ and $\sum_{i=1}^{\infty}b_i$ converges →(1)

Let E be a countable set consisting of the point $x_0, x_1, x_2,.......$

Suppose $x_n \to x_0$ as $n \to \infty$

Define $f_i(x_0)=\sum_{j=1}^{\infty}a_{ij}$ $(i=1,2,3,....)$ →(2)

$f_i(x_n)=\sum_{j=1}^{n}a_{ij}$ $(i,n=1,2,3,....)$ →(3)

$g(x)=\sum_{i=1}^{\infty}f_i(x)$ $(x\in E)$→(4)

Now we show that each f_i is continuous at x_0, for $i=1,2,3,....$

From(1), we have that $\sum_{j=1}^{\infty}a_{ij}$ converges absolutely

$\Rightarrow \sum_{j=1}^{\infty}a_{ij}$ converges

Now $\lim_{n\to\infty}f_i(x_n)=\lim_{n\to\infty}\sum_{j=1}^{n}a_{ij}$ (by(3))

$=\sum_{j=1}^{n}a_{ij}=f_i(x_0)$

$\Rightarrow \lim_{n\to\infty}f_i(x_n)=f_i(x_0)$

$\Rightarrow \lim_{x_n\to x_0}f_i(x_n)=f_i(x_0)$

$\Rightarrow f_i$ is continuous at x_0 for $i=1,2,3,....$

For each $n\ge 1$, $|f_i(x_n)|=\left|\sum_{j=1}^{n}a_{ij}\right|\le\sum_{j=1}^{n}|a_{ij}|\le\sum_{j=1}^{\infty}|a_{ij}|=b_i$

$\Rightarrow |f_i(x_n)|\le b_i \quad \forall n\ge 1$

Also, $\left|f_i(x_0)\right| \le \sum_{j=1}^{\infty}\left|a_{ij}\right| = b_i$

$\therefore \left|f_i(x_n)\right| \le b_i \quad \forall\, x \in E$

Since $\sum_{i=1}^{\infty} b_i$ converges and $\left|f_i(x_n)\right| \le b_i \ \ \forall\, i,$ then by weierstrassin-test theorem $\sum_{i=1}^{\infty} f_i$ converges uniformly on E.

Since g is the limit of the series $\sum_{i=1}^{\infty} f_i$ and $\sum_{j=1}^{\infty} f_i$ converges uniformly on E and each f_i is continuous at x_0 , by a known theorem(if $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E), g is continuous at x_0.

Since g is continuous at x_0 and $x_n \to x_0$ as $n \to \infty$, we have $\lim_{n\to\infty} g(x_n) = g(x_0)$

Consider $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n\to\infty} g(x_n)$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{\infty} a_{ij}$$

$$\therefore \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij}$$

8.3.6 Theorem (Taylors Theorem)

Suppose, the series converging in $|x| < R$. If $-R < a < R$, then f can be expanded in a power series about the point x=a which converges in $|x-a| < R-|a|$, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (|x-a| < R-|a|)$.

Proof:

suppose the series $\sum_{n=0}^{\infty} C_n x^n$ converges in $|x| < R$ and $f(x) = \sum C_n x^n \ \ (|x| < R)$.

Also, suppose $-R < a < R$ and $|x-a| < R-|a|$.

Since $\sum C_n x^n$ converges in $|x| < R$ the radius of converges of the series $\sum C_n x^n$ is greater than or equal to R we have $|x| = |x-a+a| \le |x-a| + |a| < R-|a| + |a| = R$

$\Rightarrow |x| < |x-a| + |a| < R$

So, the series $\sum_{n=0}^{\infty} |C_n| \ \ \left(|x-a| + |a|\right)^n$ converges.

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left|C_n\binom{n}{m}a^{n-m}(x-a)^m\right| \text{ converges in } |x-a|<R-|a|.$$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{n}C_n\binom{n}{m}a^{n-m}(x-a)^m \text{ converges absolutely in } |x-a|<R-|a|$$

Consider $f(x)=\sum_{n=0}^{\infty}C_nx^n=\sum_{n=0}^{\infty}C_n(x-a+a)^n$

$$=\sum_{n=0}^{\infty}C_n\sum_{m=0}^{n}\binom{n}{m}a^{n-m}(x-a)^m$$

$$=\sum_{m=0}^{n}C_n\left[\binom{n}{0}a^n(x-a)^0+\binom{n}{1}a^{n-1}(x-a)+....+\binom{n}{n}(x-a)^n\right]$$

$$=C_0+C_1\left[\binom{1}{0}a+\binom{1}{1}(x-a)\right]+C_2\left[\binom{2}{0}a^2+\binom{2}{1}a(x-a)+\binom{2}{2}(x-a)^2\right]+....+C_n\left[\binom{n}{0}a^n+\binom{n}{1}a^{n-1}(x-a)+....+\binom{n}{n}(x-a)^n\right]+.....$$

$$=\left[C_0+C_1a+C_2a^2+...+C_na^n+....\right]+(x-a)^0+C_1\left[\binom{1}{0}+\binom{2}{1}a+....+C_n\binom{n}{1}a^{n-1}+....\right](x-a)+$$

$$C_2\left[\binom{2}{2}+C_3\binom{3}{2}a+C_4\binom{4}{2}a^2+...+C_n\binom{n}{2}a^{n-2}+...\right](x-a)^2+...+$$

$$C_n\left[\binom{n}{n}+C_{n+1}\binom{n+1}{n}a+C_{n+2}\binom{n+2}{n}a^2+.....\right](x-a)^{2n}+...+....$$

$$=\sum_{n=0}^{\infty}\sum_{n=m}^{\infty}\binom{n}{m}C_na^{n-m}(x-a)^m \rightarrow(1)$$, By Know Theorem (8.21),

$f^{(m)}(a)=\sum_{n=m}^{\infty}n(n-1).......(n-m+1)C_na^{n-m}$ consider

$$\sum_{n=m}^{\infty}\binom{n}{m}C_na^{n-m}=\sum_{n=x}^{\infty}\frac{n!}{m!(n-m)!}C_na^{n-m}$$

$$=\sum_{n=m}^{\infty}\frac{1.2.....[n-(m+1)](n-m)(n-m+1).....(n-1)n}{m!\,(n-m)!}C_na^{n-m}$$

$$=\sum_{n=m}^{\infty}\frac{n(n-1)(n-2)......(n-m+1)}{m!}C_na^{n-m}$$

$$=\frac{1}{m!}f^{(m)}(a)\rightarrow(2)$$

$\therefore$ from(1) & (2), $f(x)=\sum_{m=0}^{\infty}\left\{\sum_{n=m}^{\infty}C_n\binom{n}{m}C_na^{n-m}\right\}(x-a)^m$

$$=\sum_{m=0}^{\infty}\frac{f^{(m)}(a)}{m!}(x-a)^m$$

8.3.7 Theorem:

Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converges in the segment $S=(-R,R)$.

Let E be the set of all $x \in S$ at which $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. If E has a limit point in S, then $a_n = b_n$

for n=0,1,2,….

Hence $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ holds for all $x \in S$

Proof:

Suppose that E has a limit point in S

For $n \geq 0$, put $c_n = a_n - b_n$, let $f(x)=\sum_{n=0}^{\infty} C_n x^n \ (x \in S)$

Then $f(x)=0$ on E. $\left(\because \textit{for every } x \in E, \sum a_n x^n = \sum b_n x^n, \textit{given}\right)$

Let A be the set of all limit point of E in S

Since E has a limit Point in S, then $A \neq \phi$

It is clear that A is closed in S.

Claim: A is open in S.

Let $x_0 \in A$

Then $x_0 \in S \Rightarrow |x_0| < R,$ i.e, $-R < x_0 < R$

By the Taylors theorem, $f(x)=\sum_{n=0}^{\infty} d_n (x-x_0)^n$ in $|x-x_0| < R < |x_0|$ →(1)

Now, we show that $d_n = 0$ for all n

If possible, suppose that $d_n \neq 0$ for some n.

Let K be the smallest +ve integer such that $d_n \neq 0$.

Then $d_1 = d_2 = ... = d_{k-1} = 0$.

So, from(1) $f(x) = d_K (x-x_0)^K + d_{K+1}(x-x_0)^{K+1} + ...$

$$= (x-x_0)^K \left[d_K + d_{K+1}(x-x_0) + d_{K+2}(x-x_0)^2 ...... \right]$$

$$= (x-x_0)^K g(x) \text{ where } g(x) = d_K + d_{K+1}(x-x_0) + ... \rightarrow (2)$$

Since g has the power series expansion about the point x_0, g is differentiable at x_0 and hence g is continuous at x_0

Since g is continuous at x_0 and $g(x_0) = d_K \neq 0$, then $\exists \delta > 0$ such that $g(x_0) \neq 0$ whenever

$|x-x_0| < \delta$.

It follows from(2) that $f(x)\neq 0$ whenever $0<|x-x_0|<\delta$.

That is the deleted mod $0<|x-x_0|<\delta$ dose not contain any print of E

$\Rightarrow$ x_0 is not a limit point of E, which is a contradiction to the fact that x_0 is a limit point of E

$\therefore d_n = 0$ for all n

Hence from(1), $f(x)=0$ in $|x-x_0|<R<|x_0|$

$\Rightarrow f(x)=0$ for all $x\in(x_0-\alpha,x_0+\alpha)$ where $\alpha=R'|x_0|$

$\Rightarrow(x_0-\alpha,x_0-\alpha)\subseteq E$

Let $x\in(x_0-\alpha,x_0+\alpha)$

Then $\exists\,\delta_1>0$ such that $(x_0-\delta_1,x_0+\delta_1)\subseteq(x_0-\alpha,x_0+\alpha)$

$\Rightarrow(x_0-\delta_1,x_0+\delta_1)\subseteq E$

So, every nbd of x_1 contains a point of E

($\because$ every nbd of x_1 intersects $(x_0-\delta_1,x_0+\delta_1)$)

$\Rightarrow x_1$ is a limit point of E

$\Rightarrow x_1\in A$

$\therefore(x_0-\alpha,x_0+\alpha)\subseteq A$

So, x_0 is an interior point of A

Hence, A is open

Write $B=A^c$ i.e, B is the set of all other points of S

Then B is both open and closed, $S = A\cup B$ and $A\cap B=\phi$

Since $S=(-R,R)$,S is connected

Since $S = A\cup B$,and S is connected, one of A and B must be empty

since E has a limit point, then $A=\phi$ and $B=\phi$

$\therefore S=A\cup B=A\cup\phi=A$

$\Rightarrow$ Every point of S is a limit point of E

$\Rightarrow$ E is dense in S

Since E is dense in S and $f(x)=0\ \forall\ x\in E$, and F is continuous in S, then by a known result $f(x)=0\ \forall\ x\in E$.

$\Rightarrow f^{(n)}(x) = 0 \;\; \forall \;\; x \in S$

$\Rightarrow f^{(n)}(0) = 0$

$\Rightarrow n!\, C_n = 0 \;\forall\, n$

$\Rightarrow C_n = 0 \;\; \forall\, n$

$\Rightarrow a_n = b_n \;\; \forall\, n$

8.4 SUMMARY:

This lesson focuses on helping learners comprehend power series, including derivatives and converges and apply power series properties to solve mathematical problems. High lights of this lesson definition and theorem application and examples of power series with solutions.

8.5 TECHNICAL TERMS:

- Power series
- Abels theorem
- Taylors theorem

8.6 SELF-ASSESSMENT QUESTIONS:

1. Define $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ prove that F has Derivatives of all orders at $x = 0$ and that

   $f^{(n)}(0) = 0$ for n=1,2,3,……

2. Let a_{ij} be the number in the i[th] row and j[th] column of the array

$$\begin{matrix} -1 & 0 & 0 & 0 & ---- \\ \frac{1}{2} & -1 & 0 & 0 & ---- \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & ---- \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & ---- \end{matrix}$$

so, that $a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$ prove that

$\sum_i \sum_j a_{ij} = -2, \;\; \sum_i \sum_j a_{ij} = 0.$

3. Prove that $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij},\; if \;\; a_{ij} \geq 0$ for all I and j (The case $+\infty = +\infty$ may occur)

8.7 SUGGESTED READINGS:

1. Principles of mathematical analysis by Walter Rudin, 3[rd] Edition
2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2[nd] Edition, 1985

- **Dr. L. Krishna**

LESSON- 9

THE EXPONENTIAL LOGARITHMIC AND TRIGNOMETRIC FUNCTION

OBJECTIVES:

The objective of the lesson is to understand the concepts of exponential, Logarithm and trigonometric functions. We shall derive some properties

STRUCTURE:

9.1 INTRODUCTION:

In this lesson we shall derive bexponential, logarithm and trigonometric functions and some properties such as investigation of the properties of $e^x, \log x$ and trigonometric functions.

For every complex number Z, consider the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ Since the radius of convergence of this series is ∞,then the series converges for every Complex Z

9.2 DEFINITIONS:

9.2.1 Definition

The function $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all complex number z, is called the exponential function.

9.2.2 Note

1) The function E is continuous on $(-\infty, \infty)$

2) For any complex number z and w, $E(z)E(w) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \cdot \sum_{m=0}^{\infty} \frac{w^n}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$$

$= E(z+w)$

$\therefore E(z)E(w) = E(z+w)$ for every $z, w \in C$

In general, if $z_1, z_2, ...z_n \in C$, Then $E(z_1 + z_2 + ... + z_n) = E(z_1)E(z_2), ...(z_n)$

3) $E(z) \neq 0 \ \ \forall z \in C$

4) For any $z \in C$, $E(z)E(-z) = E(z-z) = E(0) = 1$

$\therefore E(z) \neq 0 \ \ \forall z \in C$

5) $E(x) > 0$ for all real x (from definition)

6) for any +ve integer n, $x^n \to \infty \quad as \quad x \to \infty$.

$\therefore E(x) \to \infty \quad as \quad x \to \infty \ (\because E(x) = 1 + x + \frac{x^2}{2!} + ...)$

7) since $E(x)E(-x) = 1$, then $E(x) = \frac{1}{E(-x)}$ for all real x

$\therefore E(x) \to 0 \ as \ x \to \infty$

8) i) Let $x, y \in R \ni \ 0 < x < y$ by the definition of E, $E(x) < E(y) \ \to (1)$

ii) let $x, y \in R \ni \ x < y < 0$ then $0 < -y < -x$ then, by (7), $E(-y) < E(-x)$

$$\Rightarrow \frac{1}{E(y)} < \frac{1}{E(x)}$$

$$\Rightarrow E(x) < E(y)$$

$\therefore$ for any $x, y \in R \ni \ x < y,$ then $\Rightarrow E(x) < E(y)$. This shows that E is strictly increasing function on R.

9) Let $z \in C$

Consider $E'(z) = \lim\limits_{n \to 0} \frac{E(z+h) - E(z)}{h}$

$$= \lim_{n \to 0} \frac{E(z)E(h) - E(z)}{h}$$

$$= E(z) \lim_{n \to 0} \frac{E(h) - 1}{h} = E(z) \lim_{h \to 0} \left[\frac{\left(1 + h + \frac{h^2}{2!} + ..\right) - 1}{h} \right]$$

$$= E(z)$$

$$\therefore E'(z) = E(z) \ \forall \ z \in C$$

9.2.3 Definition

We have the exponential function e defined as $e=\sum_{n=0}^{\infty}\frac{1}{n!}$

9.2.4 Note

1) For any +ve integer p, $E(p)=E(1+1+....1(p\,times))$

$$=E(1)E(1)......E(1) = e.e........e, \text{ Where } e=\sum_{n=0}^{\infty}\frac{1}{n!}$$

$$=e^p$$

2) Let p>0 be any rational number then $p=\frac{n}{m}$ where m and n are +ve integers $\Rightarrow pm = n$ Then $\left(E(p)\right)^m = E(mp) = E(n) = e^n$

$$\Rightarrow E(p) = e^{\frac{n}{m}} = e^p$$

$\therefore E(p)=e^p$ for every +ve rational number 'p' →(1)

3) Let p be a -ve rational number then -p>0. Then by (1),

$E(-p)=e^{-p}$ but $E(p)E(-p)=1$

$$\Rightarrow E(p)=\frac{1}{E(-p)}=\frac{1}{e^{-p}}=e^p$$

$\therefore E(p)=e^p$ for every -ve rational number p

Hence, $E(p)=e^p$ for all rational number p

9.3 EXPONENTIAL AND LOGARITHMIC FUNCTIONS:

9.3.1 Definition

For any real number x, we define $E(x)=e^x$

9.3.2 Theorem

Let e^x be defined on R by $e^x = E(x)$ where $E(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ Then

a) e^x is continuous and differentiable for all x;

b) $(e^x)^1 = e^x$

c) e^x is a strictly increasing function of x, and $e^x > 0$;

d) $e^{x+y} = e^x e^y$

e) $e^x \to +\infty \;\; as \;\; x \to +\infty, \qquad e^x \to 0 \;\; as \;\; x \to -\infty$

f) $\lim_{x\to+\infty} x^n e^{-x} = 0$, for every n.

Proof:

a) Let $x \in R$, then $e^x = E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $n \geq 0$, write $a_n = \frac{1}{n!}$

Now $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

so, the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is ∞.

$\therefore$ by a known theorem(8.21) E(x) is continuous and differentiable for all $x \in R$ Hence, e^x is continuous and differentiable on R.

b) Now $(e^x)' = \frac{d}{dx}(e^x) = \lim_{n\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{n\to 0} \frac{e^x(e^h - 1)}{h}$

$= e^x \lim_{n\to 0} \frac{1}{h}\left(\left(1 + h + \frac{h^2}{2!} + ...\right) - 1\right)$

$= e^x \lim_{n\to 0}\left(1 + h + \frac{h}{2!} + \frac{h^3}{3!} + ...\right)$

$= e^x(1 + 0 + 0 + ...)$

$= e^x$

$\therefore (e^x)' = e^x, \ \forall \ x \in R$

c) Let $x, y \in R$ such that $0 < x < y$

$\Rightarrow \frac{x^n}{n!} < \frac{y^n}{n!}$ for every +ve integer n

So, $\sum_{n=0}^{\infty} \frac{x^n}{n!} < \sum_{n=0}^{\infty} \frac{y^n}{n!}$

$\Rightarrow E(x) < E(y)$, where ever $x < y$

$\therefore E(x)$ is a strictly increasing function and $e^x > 0$ on R

d) Consider $E(x)E(y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{y^m}{m!}$

$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$ ($\because$ By Cauchy theorem)

$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!\, x^k y^{n-k}}{k!(n-k)!}$

$=\sum_{n=0}^{\infty}\frac{1}{n!}(x+y)^n$

$=E(x+y)$

$$\therefore E(x)E(y)=E(x+y)$$

$$\Rightarrow e^{x+y}=e^x.e^y$$

e) we know that $e^x=\sum_{m=0}^{\infty}\frac{x^n}{n!}$

taking limit on both sides as $x\to\infty$, we have $\lim_{x\to\infty}e^x=\lim_{x\to\infty}\left(1+x+\frac{x^2}{2!}+.....\right)$

so, $e^x\to\infty\ as\ x\to\infty$ We have $E(x)E(-x)=1$

$$\Rightarrow E(x)=\frac{1}{E(-x)}\qquad\Rightarrow e^x=\frac{1}{e^{-x}}$$

Now, $\lim_{x\to\infty}e^x=\lim_{x\to\infty}\frac{1}{e^{-x}}=0$ so, $e^x\to 0\ as\ x\to-\infty$

f) Let n be any +ve integer

we have $e^x=E(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$\Rightarrow e^x=E(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}>\frac{x^{n+1}}{(n+1)!},\text{ for all } x>0$$

$$\Rightarrow e^{-x}<\frac{(n+1)!}{x^n.x},\text{ for all } x>0$$

$$\Rightarrow x^n e^{-x}<\frac{(n+1)!}{x},\text{ for all } x>0$$

$$\Rightarrow \lim_{x\to\infty}x^n e^{-x}\le\lim_{x\to\infty}\frac{(n+1)!}{x}=0$$

$\therefore \lim_{x\to\infty}x^n e^{-x}=0$ for all +ve integer n

9.3.3 Definition (Logarithmic Function)

Since E is a strictly increasing and differentiable function on R. it has an inverse function 'L' which is also strictly increasing and differentiable and whose domain is $E(R)$, i.e., the set of all positive number.

L is defined by $E\big(L(y)\big)=y\quad(y>0)$ or $L\big(E(x)\big)=x$ for all $x\in R$

For any $x>0$, we denote $L(x)$ by $\log x$ i.e $L(x)=\log x$

Properties of Logarithmic functions: consider $L\left(E(x)\right)=x,\ \forall\ x\in R \rightarrow (1)$

1) Differentiating on both sides w.r.to x, we have $L'\left(E(x)\right).E'(x)=1$

   $\Rightarrow L'\left(E(x)\right).E(x)=1\ \forall\ x\in R \qquad \left(\because E'(x)=E(x)\right)$

   We have for every $x\in R,\quad E(x)\geq 0$ So, $\Rightarrow L'\left(E(x)\right)=\dfrac{1}{E(x)}\ \ \forall\ x\in R$

2) Taking x=0 in (1), we have $L\left(E(0)\right)=0$b

   $\Rightarrow L(1)=0$

   $\therefore \log 1=L(1)=0$

3) By the fundamental theorem of calculus, $\int_1^y L'(x)dx=L(y)-L(1)=L(y)$

   $\therefore\ L(y)=\int_1^y L'(x)dx=\int_1^y \frac{1}{x}dx$ b

   Hence, $\log y=\int_1^y \frac{1}{x}dx$ b

4) Let $u=E(x)$ and $\vartheta=E(y)$ where $x,y\in R$ Then $u>0$ and $\vartheta>0$,

   and $L(u)=L\left(E(x)\right)=x$ and $L(\vartheta)=L\left(E(x)\right)=y$ Now,

   $L(u\vartheta)=L\left(E(x).E(y)\right)=L\left(E(x+y)\right)=x+y\ =L(u)+L(\vartheta)$

   $\therefore\ L(u\vartheta)=L(u)+L(\vartheta)$, For all $u,\vartheta\in E(R)$

5) i) Let $\in>0$ put $\delta=e^{E}$ i.e., $\delta=E(\in)$

   Then $\delta>0$ Suppose $x>\delta$. Then $L(x)>L(\delta)=L\left(E(\in)\right)=\in$

   $\Rightarrow L(x)>\in$ So, for $\in>0$, $\exists\ \delta>0 \ni\ \ x>\delta\Rightarrow L(x)>\in$

   $\therefore \lim\limits_{x\to\infty}L(x)=\infty$

   Hence, $\log x\rightarrow\infty\ as\ x\rightarrow\infty$

   ii) Let $\in>0$ then $-\in>0$ put $\delta=e^{-E}$ then $\delta>0$

   suppose $x<\delta$ then $L(x)>L(\delta)=L(e^{-\in})=L\left(E(-\in)\right)=-\in$

   $\Rightarrow L(x)<-\in$

So, for $\in > 0$, $\exists\, \delta > 0 \ni \; x > \delta \Rightarrow L(x) < -\in$ when ever $x < \delta$

$$\therefore \lim_{x\to\infty} L(x) = -\infty$$

Hence, $\log x \to -\infty \; as \; x \to 0$

6) We have $x = E\left(L(x)\right) \Rightarrow x^n = E(L(x^n)) \quad \Rightarrow x^n = E(L(x.x............x(\text{int}\, res))$

$\Rightarrow x^n = E(L(x) + L(x) + ... + L(x)) = E(nL(x))$

ll^{ly} $x^{\frac{1}{n}} = E\left(\frac{1}{n}L(x)\right) \quad \therefore x^{\alpha} = E(\alpha L(x))$ for all rational α

7) Let $\alpha > 0$

Take $\in > 0 \ni 0 < \in < \alpha \quad and \quad x > 1$

Consider $x^{-\alpha}\log x = x^{-\alpha}\int_1^x \frac{1}{t}dt = x^{-\alpha}\int_1^x t^{-1}dt$

$$< x^{-\alpha}\int_1^x t^{\in-1}dt = x^{-\alpha}\left[\frac{t^{\in}}{\in}\right]_1^x \qquad \left(\because \in > 0 \Rightarrow \in -1 > -1 \Rightarrow t^{e-1} > E^1\right)$$

$$= x^{-\alpha}\left[\frac{x^e - 1}{\in}\right] = \frac{x^{\in-\alpha} - x^{-\alpha}}{\in} < \frac{x^{\in-\alpha}}{\in}$$

$$\therefore x^{-\alpha}\log x < \frac{1}{\in x^{\alpha-\in}}; \;\; \forall \in > 0$$

Hence, $\lim_{x\to\infty} x^{-\alpha}\log x = 0 \;\; (\because 0 < \in < \alpha \Rightarrow \alpha - \in > 0)$

8) We have $x^{\alpha} = E\left(\alpha.L(x)\right)$ now $\left(x^{\alpha}\right)^1 = E'\left(\alpha.L(x)\right)\left(\alpha.L'(x)\right)$

$$= E\left(\alpha.L(x)\right)\alpha\, L'(x) = x^{\alpha}.\,\alpha.\frac{1}{x}$$

$$\therefore \left(x^{\alpha}\right)^1 = \alpha.x^{\alpha-1}$$

9.4 TRIGONOMETRIC FUNCTION:

9.4.1 Definition

For any $x \in R$, define $C(x) = \dfrac{E(ix) - E(-ix)}{2}$ and $S(x) = \dfrac{E(ix) - E(-ix)}{2i}$

9.4.2 Note:

1) Consider, $C(x) = \dfrac{E(ix) + E(-ix)}{2}$

$$= \frac{\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-ix)^n}{n!}}{2}$$

$$= \frac{\left[1 + ix + \frac{(ix)^2}{2!} + ...\right] + \left[1 + (-ix) + \frac{(-ix)^2}{2!} + ...\right]}{2}$$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right] = \cos x$$

$$\therefore C(x) = \cos x$$

2) Consider $S(x) = \frac{E(ix) - E(-ix)}{2i}$

$$= \frac{\left[1 + ix + \frac{(ix)^2}{2!} + ...\right] + \left[1 + (-ix) + \frac{(-ix)^2}{2!} + ...\right]}{2}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$= \sin x$$

$$\therefore S(x) = \sin x$$

3) Clearly $E(ix) = C(x) + iS(x)$

So, $C(x)$ and $S(x)$ are the real and imaginary parts of $E(ix)$

4) $Z \in \xi, \ E\left(\bar{z}\right) = E\left(\bar{z}\right)$

5) For any $x \in R$, $|E(ix)|^2 = E(ix).\overline{E(ix)}$

$$= E(ix)E\left(\overline{ix}\right)$$

$$= E(ix)E(-ix)$$

$$= 1$$

$$\therefore |E(ix)|^2 = 1 \ \forall \ x \in R$$

$$\Rightarrow |E(ix)| = 1 \ \forall \ x \in R$$

6) Since $E(0) = 1$, we have $C(0) = 1$ and $S(0) = 0$

7) $C'(x) = -S(x)$ and $S'(x) = C(x)$

Proof:

We have $C(x) = \dfrac{E(ix) + E(-ix)}{2}$

Differentiating on both sides w.r.to x

$$C'(x) = \frac{1}{2}\left[E'(ix)i + E'(-ix)(-i)\right]$$

$$= \frac{1}{2}\left[E'(ix)i + E'(-ix)(-i)\right]\frac{i}{i}$$

$$= (-1)\frac{E(ix) - E(-ix)}{2i}$$

$$= -S(x)$$

$$\therefore C'(x) = -S(x)$$

We have $S(x) = \dfrac{E(ix) - E(-ix)}{2i}$

Differentiating on both sides w.r.to x

$$S'(x) = \frac{1}{2i}\left[E'(ix)i - E'(-ix)(-i)\right]$$

$$= \frac{1}{2i}\left[E(ix)i - E(-ix)(-i)\right]$$

$$= \frac{E(ix) + E(-ix)}{2i}$$

$$= C(x)$$

$$\therefore S'(x) = C(x)$$

9.4.3 Definition

A function f is said to be periodic, if there is a smallest positive number τ such that $f(x+\tau) = f(x)\ \forall\ x$ in domain of f here τ is called a period of f.

9.4.4 Theorem

a) The function E is periodic, with period $2\pi i$.
b) The functions C and S are periodic, with period 2π
c) If $0 < t < 2\pi$, then $E(it) \neq 1$
d) If z is a complex number with |z|=1, there is an unique t in $[0,2\pi]$ such that $E(it) = t$

Proof:

a) First, we show that there exists a number $x > 0$ such that $C(x) = 0$

if possible, suppose that $C(x) \neq 0$ for every $x > 0 \rightarrow (1)$

clearly $C(0) = 1$

If for some elements $x_0 > 0$, $C(x_0) < 0 < 1$, then by a know result, there exists $x \in (0, x_0)$ such that $C(x) = 0$ which is a contradiction to eq(1)

$\therefore C(x) > 0$ for all $x > 0$

$\Rightarrow S'(x) > 0$ for all $x > 0$ $\quad (\because S'(x) = C'(x)\ \forall\ x)$

$\Rightarrow S$ is strictly monotonically increasing function for $x > 0$

So, $S(x) > S(0) = 0$ for all $x > 0$

$\Rightarrow S(x) > 0$ for all $x > 0$

Hence, if $0 < x < y$, we have $(y - x)S(x) < \int_x^y S(t)\, dt = \int_x^y -C'(t)\, dt$

$= [-C(t)]_x^y = C(x) - C(y) \leq 1 + 1 = 2$

$\Rightarrow (y - x)S(x) < 2, \forall\, y > x$

$\Rightarrow y < x + \dfrac{2}{S(x)}$ $\forall\ y > x$, which can't be true for large y , since $S(x) > 0$

$\therefore$ There exists $x > 0$ such that $C(x) = 0$

let x_0 be the smallest positive number such that $C(x_0) = 0$

this exists, since the set of 3ems of a continuous function is closed and $C(0) \neq 0$

define $\pi = 2x_0$. Then $C(y) \neq 0$ for all $y < x_0$

$C\left(\dfrac{\pi}{2}\right) = 0$ and hence $S\left(\dfrac{\pi}{2}\right) = \pm 1$

Since $C(x) > 0$ in $\left(0, \dfrac{\pi}{2}\right)$, S is increasing in $\left(0, \dfrac{\pi}{2}\right)$ and hence $S\left(\dfrac{\pi}{2}\right) = 1$

$\therefore E\left(\dfrac{i\pi}{2}\right) = C\left(\dfrac{\pi}{2}\right) + iS\left(\dfrac{\pi}{2}\right) = 0 + i.1 = i$

So, $E(i\pi) = E\left(\dfrac{i\pi}{2} + \dfrac{i\pi}{2}\right) = E\left(\dfrac{i\pi}{2}\right)E\left(\dfrac{i\pi}{2}\right) = i.i = -1$ and

$E(2\pi i) = E(i\pi + i\pi) = E(i\pi)E(i\pi) = (-1)(-1) = 1$

$\therefore E(z+2i\pi)=E(z)E(2i\pi)=E(z)$ for all complex number z

Hence, E is a periodic function, with period $2i\pi$

b) For any real x, consider $C(x+2\pi)=\frac{1}{2}\left[E\left(i(x+2\pi)\right)+E\left(-i(x+2\pi)\right)\right]$

$=\frac{1}{2}\left[E\left(ix\right)+E\left(-ix\right).E(-2i\pi)\right]$ ($\because$ E is period with period $2i\pi$)

$=\frac{1}{2}\left[E\left(ix\right)+E\left(-ix\right)\right]=C(x)$

And $S(x+2\pi)=\frac{1}{2i}\left[E\left(i(x+2\pi)\right)-E\left(-i(x+2\pi)\right)\right]$

$=\frac{1}{2i}\left[E\left(ix\right)-E\left(-ix\right)\right]$

$=S(x)$

$\therefore$ C and S are periodic with period 2π

c) suppose $0<t<\frac{\pi}{2}$ and $E(it)=x+iy$ where $x,y\in R$ $\Rightarrow E(-it)=x-iy$

$\therefore 1=E(0)=E(it-it)=E(it)E(-it)=(x+iy)(x-iy)$

$\Rightarrow 1=x^2+y^2 \rightarrow (1)$

$\Rightarrow 0<x<1 \;\; and \;\; 0<y<1$

Consider $E(4it)=E(it+it+it+it)$

$=E(it)E(it)E(it)E(it)$

$=(x+iy)^4$

$=x^4+y^4-6x^2y^2+4ixy(x^2-y^2)$

If $E(4it)$ is a real no., then $4xy(x^2-y^2)=0 \rightarrow (2)$

As $x\neq 0\,\&\,y\neq 0$, $xy\neq 0$

So, from(2) , $x^2-y^2=0$

$\Rightarrow x^2=y^2$

Hence , from(1), $x=\frac{1}{\sqrt{2}}$ and $y=\frac{1}{\sqrt{2}}$

$$\therefore E(i\,4t)=\left(\frac{1}{\sqrt{2}}\right)^4+\left(\frac{1}{\sqrt{2}}\right)^4-6\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2 \quad =-1 \quad \rightarrow (3)$$

We have $0 < t < \frac{\pi}{2} \Rightarrow 0 < 4t < 2\pi$

From(3), $E(it_1) = -1$ where $t_1 = 4t \Rightarrow E(it_1) \neq 1$ *if* $0 < t_1 < 2\pi$

Hence , $0 < t < 2\pi \quad \Rightarrow E(it) \neq 1$

d) Fix a complex number z such that $|z| = 1$

Let $z = x + iy$ where $x, y \in R$

As $|z| = 1, x^2 + y^2 = 1$

Case i: suppose $x \geq 0$ of $y \geq 0$

Since 'C' is decreases on $(0, \frac{\pi}{2})$ from 1 to 0, by a known result, $C(t) = x$ for some $t \in (0, \frac{\pi}{2})$.

Since $C^2 + S^2 = 1$ and $S \geq 0$ on $(0, \frac{\pi}{2})$, we get $S(t) = y$ for some $t \in (0, \frac{\pi}{2})$.

Therefore $E(it) = C(t) + iS(t) = x + iy = z$ for some $t \in \left(0, \frac{\pi}{2}\right) \subseteq [0, 2\pi]$

Case ii: Suppose $x < 0$ and $y \geq 0$

Then $-x > 0$ and $y \geq 0$, we have $z = x + iy$

So, $-iz = -ix + y$ that implies $-iz = y + ix_1$, where $x_1 = -x > 0$

Clearly, $|-iz| = \sqrt{y^2 + x_1^2} = \sqrt{y^2 + (-x)^2} = |z| = 1$

So, by Case (i), $E(it) = -iz$ for some $t \in (0, \frac{\pi}{2})$.

That implies $z = \frac{-1}{i} E(it) = iE(it) = E\left(\frac{i\pi}{2}\right) E(it) = E\left(it + \frac{i\pi}{2}\right) \leq 1$

$$= E\left(i\left(t + \frac{\pi}{2}\right)\right)$$

$$= E(it_1)$$

where $t_1 = t + \frac{\pi}{2} \in \left[\frac{\pi}{2}, \pi\right] \subseteq [0, 2\pi]$

Case iii: Suppose $y < 0$ and $x > 0$

Then $x > 0$ and $-y > 0$, we have $z = x + iy$

That implies $-z = x + i(-y)$

$$= x + iy_1 \text{ where } y_1 = -y > 0$$

Now, $|-z| = |z| = 1$

So, by Case (ii), we get $t \in (0, \frac{\pi}{2})$ such that $E(it) = -z$

That implies $z = -E(it) = E(i\pi)E(it) = E(i\pi + it)$

$$= E\big(i(\pi + t)\big) = E(it_2)$$

Where $t_2 = \pi + t \in (0, 2\pi)$

Therfore $z = E(it)$ for some $t \in (0, 2\pi)$

Case iv: Suppose $x < 0$ and $y < 0$, then $-x > 0$ and $-y > 0$

We have $z = x + iy$

That implies $-z = -x + i(-y) = x_1 + iy_1$

Where $x_1 = -x > 0$ and $y_1 = -y > 0$

Now $|-z| = |z| = 1$

So, by Case (i), we get $t \in (0, \frac{\pi}{2})$ such that $E(it) = -z$

That implies $z = -E(it) = E(i\pi)E(it) = E(i(\pi + t))$

$$= E(it_3)$$

Where $t_3 = t + \pi \in [0,2\pi]$

Therefore $E(it) = z$ for some $t \in [0, 2\pi]$

9.5 SUMMARY:

This lesson is designed to introduce learners to the fundamental concepts of the Exponential, Logarithmic and Trigonometric, exploring their properties and applying them to real-world contexts. This lesson provides a solid foundation for learners to develop their understanding. Key takeaways of this lesson are definitions and theorems, applications of the Exponential, Logarithmic and Trigonometric in mathematical and real-world problems and examples and exercises to their force understanding.

9.6 TECHNICAL TERMS:

- Exponential functions
- Logarithemic functions
- Trigonometric functions
- Periodic functions

9.7 SELF-ASSESSMENT QUESTION:

1. Prove the following limit relations

a. $\lim_{x \to 0} \frac{b^x - 1}{x} = \log b \quad (b > 0)$

b. $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

c. $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

d. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

2. Find the following limits.

a. $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}$

b. $\lim_{x\to\infty}\frac{x}{\log x}\left(n^{1/x}-1\right)$

c. $\lim_{x\to 0}\frac{\tan x - x}{x(1-\log x)}$

d. $\lim_{x\to 0}\frac{x-\sin x}{\tan x - x}$

3. Suppose $f(x)f(y)=f(x+y)$ for all real x and y

a. Assuming that f is differentiable and non-zero Prove that $f(x)=e^{cx}$, where c is a constant

b. Prove That something, assuming only that f is continuous

4. If $0 < x < \frac{\pi}{2}$, Prove that $\frac{\pi}{2} < \frac{\sin x}{x} < 1$

5. For $n = 0,1,2,3.....$, and x real, Prove That $|\sin(nx)| \le n|\sin x|$

9.8 SUGGESTED READINGS:

1) Principles of mathematical analysis by Walter Rudin, 3rd Edition

2) Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985

- **Dr. L Krishna**

LESSON- 10
LINEAR TRANSFORMATIONS

OBJECTIVES:

The objective of this unit is to explore the concepts of linear transformations and their applications in advanced mathematical analysis.
It aims to

1. Understand the principles of linear transformations and their role in differentiating functions
2. Analyse the contraction principle and its significance in fixed-point theory.
3. Examine the inverse function theorem and its applications in multivariable calculus.
4. Develop problem-solving skills related to differentiability and transformations in higher dimensions.

STRUCTURE:

10.1 INTRODUCTION:

This lesson starts with an exploration of sets of vectors in Euclidean R^nspace. While the algebraic principles discussed here apply to any finite-dimensional vector space over any field of scalars, we will focus on the familiar framework of Euclidean spaces for simplicity.

10.2 DEFINITIONS:

1. A nonempty set $X \subseteq R^n$ is a vector space over R if
   i. $\bar{x} + \bar{y} \in X$ and
   ii. $c\bar{x} \in X$ for all $\bar{x}, \bar{y} \in X$ and for all scalars c

2. If $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k \in R^n$ and $c_1, c_2, ...c_k$ are scalars then

$c_1\bar{x}_1 + c_2\bar{x}_2 + ... + c_k\bar{x}_k$ Is called a linear combination of $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$.

3. If $S \subseteq R^n$ and if E is the set of all linear combinations of elements of S

We say that S spans E (or) that E is the span of S.

10.2.1 Note: - Every span is a vector space.

4. A set consisting of vectors $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ is said to be linearly dependent if there exists a scalars $c_1, c_2, ...c_k$,not all zero, such that $c_1\bar{x}_1 + c_2\bar{x}_2 + ... + c_k\bar{x}_k = \bar{0}$

5. A set consisting of vectors $\bar{x}_1, \bar{x}_2, ..., \bar{x}_k$ is said to be linearly independent if there exists a scalars $c_1, c_2, ...c_k$,not all zero, such that $c_1\bar{x}_1 + c_2\bar{x}_2 + ... + c_k\bar{x}_k = \bar{0}$ implies $c_1 = c_2 =, ...c_k = 0$.

10.2.2 Note: Observe that no independent set contains the null vector.

6. If a vector space X contains an independent set of $'r'$ vectors but does not contain independent set of $r + 1$ vectors, then we say that X has dimension r, and we write $\dim X = r$.

10.2.3 Note: The set consisting of $\bar{0}$ alone is a vector space; its dimension is 0.

7. Let X be a vector space. A subset B of X is called a basis of X if
   i. B is linearly independent and
   ii. B spans X

10.2.4 Note 1. Observe that if $B = \{\bar{x}_1, \bar{x}_2, ..., \bar{x}_k.\}$ is a basis of X, then every element $\bar{x}$ in X has a unique representation of the form $\bar{x} = \sum_{i=1}^{K} c_i\bar{x}_i$ the numbers $c_1, c_2, ...c_k$ are called the coordinates of $\bar{x}$ with respect to the basis B.

10.2.5 Note 2. Consider the vector space R^n.

The set $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ where $\bar{e}_j$ is the vector in R^n whose jth coordinate is 1

And whose other coordinates are all 0, is a basis for R^n

This basis is called as the standard basis of R^n

10.3 THEOREMS ON LINEAR TRANSFORMATIONS:

10.3.1 Theorem Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then $\dim X \le r$.

Proof: Given that r is a positive integer

Suppose $X$is a vector space, spanned by a set S of vectors.

Let the r vectors of S be$\bar{x}_1, \dots, \bar{x}_r$.Then

$s = \{\overline{x_1}, \overline{x_2}, \dots, \overline{x_r}\}$.

Claim: dim$X \le r$

If possible suppose that dim $X > r$

Then we get a linearly independent set, say $Q = \{\bar{y}_1, \dots, \bar{y}_r, \bar{y}_{r+1}\}$of $r + 1$ vectors in X.

Since $\bar{y}_1 \in X$and S spans$X$, we have that $\bar{y}_1$ is a linear combination of elements of S …..(1)

So, $S \cup \{\bar{y}_1\}$ is a linear dependent set in X.

Write $S_1 = \{\bar{y}_1, \overline{x_1}, \dots, \bar{x}_r\}$

Then S_1 is linearly dependent and S_1 spans$X$.

By (1) $\exists$ scalars $b_1, b_2, b_3, \dots .. b_r$ such that $\bar{y}_1 + b_1\bar{x}_1 + \cdots + b_r\bar{x}_r = \overline{0}$ …..(2)

If allb_i's are zero, then$\bar{y}_1 = 0$,

So, some b_k ,$1 \le k \le r$ is non zero.

Therefore from (2), $\bar{x}_k$is a linear combination of $\bar{x}_1, \cdots, \bar{x}_{k-1}, \bar{x}_{k+1}, \cdots, \bar{x}_r$ and $\bar{y}_1$ ….. (3)

Write $s_2 = \{\bar{y}_1, \bar{x}_1, \dots, \bar{x}_{k-1}, \bar{x}_{k+1}, \dots, \bar{x}_r\}$.

Now we prove that S_2 spans$X$.

Let $\bar{u} \in X$.

Since S_1 spans$X$,we have

$$\overline{u} = \sum_{i=1}^{r} c_i \overline{x}_i + b_1 \overline{y}_1 \quad \text{….. (4)}$$

By (3)$\bar{x}_k = d_1\bar{y}_1 + a_1\bar{x}_1 + \cdots + a_{k-1}\bar{x}_{k-1} + a_{k+1}\bar{x}_{k+1} + \cdots + a_r\bar{x}_r$.

Where$a_1, \dots a_{k-1}, a_{k+1}, \dots, a_r$ and d_1 are some scalars.

So, from (4) we get

$$\bar{u} = c_1\bar{x}_1 + \cdots + c_{k-1}\bar{x}_{k-1} + c_k\big(d_1\bar{y}_1 + a_1\bar{x}_1 + \cdots + a_{k-1}\bar{x}_{k-1} + a_{k+1}\bar{\bar{x}}_{k+1} + \cdots a_r\bar{x}_r\big) + c_{k+1}\bar{x}_{k+1} + \dots + c_r\bar{x}_r + b_1\bar{y}_1.$$

$$\Rightarrow \bar{u} = (c_1 + a_1c_k)\bar{x}_1 + \cdots + (c_{k-1} + a_{k-1}c_k)\bar{x}_{k-1} + (c_{k+1} + a_{k+1}c_k)\bar{x}_{k+1} + \cdots + (c_r + a_rc_k)\bar{x}_r + +(b_1 + d_1c_k)\bar{y}_1.$$

$\Rightarrow$ $\bar{u}$is a linear combination of $\bar{x}_1, \dots, \bar{x}_{k-1}, \bar{x}_{k+1}, \cdots, \bar{x}_r$ and $\bar{y}_1$.

$\therefore$ S_2 spans X.

Since $\bar{y}_2 \in X$ and S_2 spans x, we have that $\bar{y}_2$ is a linear combination of vectors ofS_2

$\Rightarrow$ $S_2 \cup \{\bar{y}_2\}$ is linearly dependent.

Write $S_3 = S_2 \cup \{\bar{y}_2\}, = \{\bar{y}_1, \bar{y}_2, \bar{x}_1, \dots, \bar{x}_{k-1}, \bar{x}_{k+1}, \cdots, \overline{x_r}\}$

Then S_3 is a linearly dependent set in$X$.

It is clear that $\exists x_j$ in $S_3\backslash\{\bar{y}_1,\overline{y_2}\}\exists x_j$ is a linear combination of$\bar{y}_1,\bar{y}_2,\bar{x}_1,\dots,\bar{x}_{j-1},\bar{x}_{j+1},\dots,x_r$.

Write $S_4 = \{\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{x}_1,\dots,\bar{x}_{j-1},\bar{x}_{j+1},\dots,\bar{x}_{k-1},\bar{x}_{k+1},\dots,\bar{x}_r\}$.

Clearly, S_4 spans$X$.

Proceeding like this, after r steps, we get ar set $\{\bar{y}_1,\dots,\bar{y}_r\}$ and $\bar{y}_{r+1}$ is a linear combination of $\bar{y}_1,\bar{y}_2,\dots,\bar{y}_r$.

So, $Q = \{\bar{y}_1,\dots,\bar{y}_r,\bar{y}_{r+1}\}$ is linearly dependent.

$\therefore$ dim$X \leq r$

10.3.2 Corollary: dim $R^n = n$.

Proof: since $\{\bar{e}_1,\bar{e}_2,\dots,\bar{e}_n\}$ spans R^n, by the above Theorem, $\dim(\mathrm{R}^n) \leqslant n$ $\rightarrow (1)$

since $\{\bar{e}_1,\bar{e}_2,\dots,\bar{e}_n\}$ is a L .I set in R^n , $\dim(R^n) \geq n$ $\rightarrow (2)$

From (1) & (2), dim$R^n = n$.

10.3.3 Theorem Suppose X is a vector space, and dim$X = n$.

(a) A set E of n vectors in X spans X if and only if E is independent.

(b) X has a basis, and every basis consists of n vectors.

(c) If $1 \leq r \leq n$ and $\{\bar{y}_1,\dots,\bar{y}_r\}$ is an independent set in X, then X has a basis Containing$\{\bar{y}_1,\dots,\bar{y}_r\}$.

Proof: Given, X is a Vector spaceanddim $X = n$.

a) Let$E = \{\bar{x}_1,\bar{x}_2,\dots,\bar{x}_n\} \subseteq X$.

$$X = \langle E\rangle$$

Suppose Espans$X$.

Now we prove that E is linearly independent in X.

If possible suppose that E is linearly dependent.

Then $\exists$scalars$\alpha_1,\alpha_2,\cdots\alpha_n$, not all zero such that

$$\alpha_1\bar{x}_1 + \alpha_2\bar{x}_2 + \cdots + \alpha_n\bar{x}_n = 0$$

Suppose $\alpha_k \neq 0$ for some$1 \leq K \leq n$.

Then $\bar{x}_k$ is a linear combination of $\bar{x}_1,\dots,\bar{x}_{k-1},\bar{x}_{k+1},\dots,\bar{x}_n$.

$\therefore \{\bar{x}_1,\dots,\bar{x}_{k-1},\bar{x}_{k+1},\dots,\bar{x}_n\}$Spans X

So, by the above Th., dim$X \leq n-1$.

$\Rightarrow n \leq n-1$This is not possible.

$\therefore E$is a linearly independent set in X.

Conversely, suppose that$E$ is linearly independent.

Set of n vectors, we have that.

Sincedim $X = n$, and E is linearly independent X,

For any $\bar{y} \in x, E \cup \{\bar{y}\}$ is a linearly dependent set in X.

Let$\bar{y} \in X$.

Then $EU\ \{\bar{y}\}$ is linearly dependent.

so, $\exists$ scalars $a_1, \ldots, a_n, a_{n+1}$ not all zero,

such that $a_1\bar{x}_1 + \cdots + a_n\bar{x}_n + a_{n+1}\bar{y} = \bar{0}$ $\rightarrow (1)$

If $a_{n+1} = 0$, then $a_1\bar{x}_1 + \cdots + a_n\bar{x}_n = \bar{0}$.

$\Rightarrow a_i = 0\ \forall\ 1 \le i \le n$ ($\because E$ is L.I.)

This is a Contradiction.

$\therefore a_{n+1} \neq 0$.

So, from (1), $\bar{y}$ is a linear combinationof$\overline{x_1}, \ldots, \bar{x}_n$.

$\therefore E$ Spans X.

b) Since dim $X = n$,

X Contains alinearly independentSet$B$of n vectors and does not containonly linearly independent Set of $n + 1$ vectors.

By part (a), B spans x.

Hence, B is a basis of Xcontaining n vectors.

Let ' S ' be any basis of ' X ' containing m eles.

Then by the above Th., dim$x \le m$ $\rightarrow (1)$

since s is a linearly independent set of $m$vectors, by the definition of dim $X, m \le n$ $\rightarrow (2)$

From (1) & (2), $m = n$.

Hence, any basis of X contains n vectors.

C) Suppose, $1 \le r \le n$, and $\{\bar{y}_1, \ldots, \bar{y}_r\}$ is a, linearly independent set in X.

Sincedim$X = n$, we get a basis $\{\bar{x}_1, \ldots, \bar{x}_n\}$ of n vectors in X.

Write$S = \{\bar{y}_1, \ldots, \bar{y}_r, \bar{x}_1, \ldots, \bar{x}_n\}$

clearly, Sspans$X$.

Also, S is linearly dependentin X.

So, one of the vectors,say$\bar{x}_i$ is a linear combination of $\bar{y}_1, \ldots, \bar{y}_r, \bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i+1}, \ldots, \bar{x}_i$

Remove $\bar{x}_i$ from$S$.

Then the set $S' = S \setminus \{\bar{x}_i\}$ still spans X.

If we repeat this process r times, by (a),

We get a basis of X which Contains$\{\bar{y}_1, \cdots, \bar{y}_r\}$.

10.3.4 Definition: A mapping A from a vector space X into a vector space Y is said to be a linear transformation if (i) $A(\mathrm{x}_1 + \mathrm{x}_2) = A\mathrm{x}_1 + A\mathrm{x}_2$

(ii) $A(c\mathrm{x}) = cA\mathrm{x}$

for all $\mathrm{x}, \mathrm{x}_1, \mathrm{x}_2 \in X$ and all scalars c.

10.3.4.1 Note: Set $A: X \to Y$ be a linear transformation.

i) $A(0) = O_Y$.

ii) we write AX instead of $A(x)$.

10.3.5 Definition: A linear transformation of a vector spaceX into itself is called a linear operator on X.

10.3.6 Definition: - Let X be a vector space. A linear operator A on X is Said to be invertible if

i)$A$is one-one, and

ii) A maps Xonto X.

10.3.6.1 Note: -If A is a linear operator on X, then A^{-1} is an operator on X, defined by $A^{-1}\big(A(x)\big) = x \forall x \in X$.

10.3.6.2 Note: $A\big(A^{-1}(\mathrm{x})\big) = x\ for\ all\ x \in X.$

Proof: Let $x \in X$.

since$A$ is onto, $\exists y \in x$ such that $A(y) = x$.

By the def of A^{-1}, $y = \bar{A}^{-1}\big(A(y)\big) = A^{-1}(x)$

$$\Rightarrow A(y) = A\big(A^{-1}(x)\big)$$
$$\Rightarrow x = A\big(A^{-1}(x)\big)$$

10.3.6.3 Note: A^{-1} is a linear operator on X.

Proof: we have A is a linear operator on X.

Let $x_1, x_2 \in X$.

since A maps X onto $X, \exists y_1, y_2 \in X$ such that $A(y_1) = x_1, \& A(y_2) = x_2$.

$\Rightarrow A^{-1}(A(y_1)) = A^{-1}(x_1)$ and $A^{-1}(A(y_2)) = A^{-1}(x_2)$.
$\Rightarrow y_1 = A^{-1}(x_1)$ ad $y_2 = A^{-1}(x_2)$.

$$\begin{aligned} Now\ \ A^{-1}(x_1+x_2) &= A^{-1}(A(y_1)+A(y_2)) \\ &= A^{-1}(A(y_1+y_2)) \\ &= y_1+y_2 \\ &= A^{-1}(x_1)+A^{-1}(x_2). \end{aligned}$$

Let $x \in X$ and let ' c ' be a scalar.

since A maps X onto $X, \exists\ y \in X \ni A(y) = x$.

So, $y = A^{-1}(A(y)) = A^{-1}(x)$

Now $A^{-1}(cx) = A^{-1}(cA(y)) = A^{-1}(A(cy)) = cy = cA^{-1}(x)$.

Hence, A^{-1} is a linear operator on$X$.

10.3.7 Theorem A linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X.

Proof: Let A be a linear operator on a finite dimensional vector space$X$.

Since X is a finite dimensional vector space,

We get a basis, say $B = \{\bar{x}_1, \overline{x_2}, \dots, \overline{x_n}\}$ of X.

Consider $R(A)$, the range of$A$.

Write $Q = \{A\overline{x_1}, A\bar{x}_2, \dots, A\overline{x_n}\}$.

Then $Q \subseteq R(A)$

Now we Show that Q spans $R(A)$.

Let $\bar{y} \in R(A) \Rightarrow \bar{y} = A(\bar{x})$ for some $\bar{x} \in X$

Since B spans X and $\bar{x} \in X$, we have that

$\bar{x} = \alpha_1\bar{x}_1 + \alpha_2\bar{x}_2 + \cdots. + \alpha_n\bar{x}_n$ for some scalars $\alpha_1, \alpha_2, \dots, \alpha_n$.

So, $\bar{y} = A(\bar{x})$

$$\begin{aligned} &= A(\alpha_1\bar{x}_1 + \alpha_2\overline{x_2} + \cdots + \alpha_n\bar{x}_n) \\ &= \alpha_1 A(\bar{x}_1) + \alpha_2 A(\bar{x}_2) + \cdots + \alpha_n A(\bar{x}_n) \end{aligned}$$

$\therefore$ $\bar{y}$ is the linear combination of vectors of Q

$\therefore$ Q spans $R(A)$ → (1)

We know that a set E of $n$vectors in X spans X iff$E$ is independent (put (a) of Th-4.3)

Span of $Q = X$

So, we have $R(A) = x$ iff Q is independent → (2)

From (1) &(2)

So, it is enough if we show that Q is independent iff$A$ is $1-1$.

Suppose A is $1-1$.

Let $c_1, c_2, \ldots, c_n$ be scalars $\ni$

$c_1 A(\bar{x}_1) + C_2 A(\bar{x}_2) + \cdots + C_n A(\bar{x}_n) = \overline{0}$
$\Rightarrow A(C_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) = \overline{0}$
$\Rightarrow c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n = \overline{0}$($\because$ A is $1-1$)
$\Rightarrow C_1 = C_2 = . = C_n = 0 \cdot$ ($\because$ since$B$ is independent)
$\therefore Q$ is independent .

Conversely, suppose that Q is independent.

Let $\bar{x} \in X \ni A(\bar{x}) = \overline{0}$ $\rightarrow (3)$

Since Bspans X, and $\bar{x} \in X$,

$$\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n \rightarrow (4)$$

for some scalars $c_1, c_2, \ldots, c_n$.

So, (3) $\Rightarrow A(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) = \overline{0}$

$c_1 A(\bar{x}_1) + c_2 A(\bar{x}_2) + \cdots + C_n A(\bar{x}_n) = \overline{0}$
$\Rightarrow c_1 = c_2 = \cdots = c_n = 0$ (Q is $L.$I) $\rightarrow (5)$

So, from (4) & (5), we get $\bar{x} = \overline{0}$.

Hence the Theorem follows.

Suppose $A(\bar{x}) = A(\bar{y})$where $\bar{x}, \bar{y} \in X$.

$\Rightarrow A(\bar{x}) - A(\bar{y}) = \overline{0}$.

$\Rightarrow A(\bar{x} - \bar{y}) = \overline{0}$

$\Rightarrow \bar{x} - \bar{y} = 0 \Rightarrow \bar{x} = \bar{y}$

$\therefore A$ is 1-1

10.3.7.1 Notations

Let X, Y be two vector spaces.

i) The set of all linear transformations of X into Y is denoted by $L(X, Y)$.

ii) We simply write $L(X)$ instead of $L(\dot{X}, X)$.

10.3.8 Definition-Cum-Remark:

Let $A_1, A_2 \in L(X, Y)$ and c_1, c_2 be scalars.

For any $\bar{x} \in X$, define $(c_1A_1 + c_2A_2)(\bar{x}) = c_1A_1(\bar{x}) + c_2A_2(\bar{x})$.

Then $c_1A_1 + c_2A_2 \in L(X, Y)$

Proof: Let $\overline{x_1}, \bar{x}_2 \in X$ and let α, β be scalars.

Consider $(c_1A_1 + c_2A_2)(\alpha\overline{x_1} + \beta\overline{x_2}) =$

$$= c_1A_1(\alpha\bar{x}_1 + \beta\overline{x_2}) + c_2A_2(\alpha\overline{x_1} + \beta\overline{x_2})$$
$$= c_1[A_1(\alpha\bar{x}_1) + A_1(\beta\bar{x}_2)] + c_2[A_2(\alpha\bar{x}_1) + A_2(\beta\overline{x_2})]$$
$$= c_1[\alpha A_1(\bar{x}_1) + \beta A_1(\overline{x_2})] + c_2[2A_2(\overline{x_1}) + \beta A_2(\overline{x_2})]$$
$$= [c_1\alpha A_1(\overline{x_1}) + c_1\beta A_1(\bar{x}_2)] + [c_2\alpha A_2(\bar{x}_1) + c_2\beta A_2(\overline{x_2})]$$
$$= [\alpha(c_1A_1)(\bar{x}_1) + \beta(c_1A_1)(\overline{x_2})] + [\alpha(c_2A_2)(\bar{x}_1) + \beta(c_2A_2)(\overline{x_2})]$$
$$= \alpha(c_1A_1 + c_2A_2)(\overline{x_1}) + \beta(c_1A_1 + c_2A_2)(\overline{x_2}).$$
$\therefore c_1A_1 + c_2A_2 \in L(X, Y)$.

10.3.9 Definition-Cum-Remark:

Let X, Y and Z be vector spaces. If $A \in L(X, Y)$ and $B \in L(Y, Z)$, we define their product BA to be the composition of AB and BA. i ·e, for any $\bar{x} \in \dot{X}, (BA)(\bar{x}) = B\big(A(\bar{x})\big)$. Then $BA \in L(x, z)$.

Proof: Let $\bar{x}, \bar{y} \in X$ and let α, β be scalars

Consider $BA(\alpha\bar{x} + \beta\bar{y}) = B\big(A(\alpha\bar{x} + \beta\bar{y})\big)$

$$= B[A(\alpha\bar{x}) + A(\beta\bar{y})]$$
$$= B[\alpha A(\bar{x}) + \beta A(\bar{y})]$$
$$= B\big(\alpha A(\bar{x})\big) + B\big(\beta A(\bar{y})\big)$$
$$= \alpha B\big(A(\bar{x})\big) + \beta B\big(A(\bar{y})\big)$$
$$= \alpha(BA)(\bar{x}) + \beta(BA)(\bar{y})$$
$\therefore BA \in L(X, Z)$.

Note: -BA need not be the some as AB even if $X = Y = Z$.

10.3.10 Definition: For $A \in L(\mathrm{R}^n, R^m)$, define the norm $\|A\|$ of A as the supremum of all numbers$|A\bar{x}|$, where $\bar{x}$ ranges over all vectors in $R^n$with $|\bar{x}| \leqslant 1$.

$$i.e \ \ \|A\| = \sup_{\substack{x\in\mathbb{R} \\ |\bar{x}|\leqslant 1}} |A\bar{x}|$$

Observation (1): Let $A \in L(\mathrm{R}^n, R^m)$then$|A\bar{x}| \leq \|A\||\bar{x}| \forall \bar{x} \in \mathrm{R}^n$.

Proof:

Let$\bar{x} \in \mathrm{R}^n$.

If$\bar{x} = \overline{0}$, then clearly$|A\bar{x}| \leq \|A\||\bar{x}|$.

Suppose$\bar{x} \neq \overline{0}$. Put$\bar{y} = \frac{\bar{x}}{|\bar{x}||}$then$|\bar{y}| = 1$.

By the def of $\|A\|, \|A(y)\|' \leq \|A\|$

$$\Rightarrow \left|A\left(\frac{\bar{x}}{|\ \bar{x}\ |}\right)\right| \leq \|A\| \Rightarrow \frac{1}{|\bar{x}|}|A\bar{x}| \leq \|A\| \cdot (\because A \text{ is linear})$$
$$\Rightarrow |A\bar{x}| \leq \|A\||\bar{x}|$$

Hence,

$|A\bar{x}| \leq \|A\||\bar{x}| \forall \bar{x} \in \mathrm{R}^n$.

2) If λ is such that $|A\bar{x}| \leq \lambda|\bar{x}| \forall \bar{x} \in \mathrm{R}^n$, then $||A|| \leq \lambda$.

Proof: Let $\bar{x} \in \mathrm{R}^n |\bar{x}| \leq 1$.

Suppose $|A\bar{x}| \leq \lambda|\bar{x}| \leq \lambda$.

$\therefore \lambda$ is an upper bound of $|A\bar{x}|$, $\bar{x} \in \mathrm{R}^n |\bar{x}| \leq 1$.

Since

$||A||$ is the supremum of λ, then

$||A|| \leq \lambda$

10.3.11 Theorem

(a) If $A \in L(R^n, R^m)$, then $||A|| < \infty$ and A is a uniformly continuous mapping of R^n into R^m.

(b) If $A, B \in L(R^n, R^m)$ and c is a scalar, then

$$||A + B|| \leq ||A|| + ||B||, ||cA|| = |c|||A||$$

With the distance between A and B defined as $||A - B||, L(R^n, R^m)$ is a metric space.

(c) If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$, then

$||BA|| \leq ||B||||A||$.

Proof:

(a) Suppose $A \in L(R^n, R^m)$

Let $E = \{\bar{e}_1, \ldots\ldots \overline{e_n}\}$ be the standard basis of R^n.

Let $\bar{x} \in R^n \ni |\bar{x}| \leq 1$.

Since E spans R^n, we get some scalars $c_1 \to c_n$ such that

$\bar{x} = c_1\overline{e_1} + \cdots + c_n\overline{e_n}$.

$|z_1 + z_2| \leq 1$.

we have $|\bar{x}| \leqslant 1$.

$\Rightarrow |c_1\bar{e}_1 + c_n\bar{e}_n| \leq 1 \Rightarrow |(c_1, c_2, \ldots\ldots c_n)| \leq 1$

$\Rightarrow \sqrt{c_1^2 + \cdots + c_n^2} \leq 1 \Rightarrow |c_i| \leq 1 \quad \forall \ 1 \leq i \leq \mathrm{n}$

Now consider

$$|A\bar{x}| = |A(c_1\bar{e}_1 + \cdots + c_n\bar{e}_n)| = |c_1A(\bar{e}_1) + \cdots + c_nA(\bar{e}_n)|.$$

$$\leq |c_1||A(\bar{e}_1)| + \cdots + |c_n||A(\bar{e}_n)|$$

$$\leq |A(\bar{e}_1)| + \cdots + |A(\bar{e}_n|)(\because |c_i| \leq 1 \forall i)$$

$$< \infty$$

$\therefore$For every $\bar{x} \in R^n$ with $|\bar{x}| \leqslant 1; |A\bar{x}| < \infty$

Hence,

$||\mathrm{A}|| = \sup\{|A\bar{x}|/|x| \leq 1\} < \infty$,

$$\text{i.e} \quad \|A\| < \infty.$$

Let $\epsilon > 0$.

If $A = \overline{0}$, then clearly A is uniformly continuous.

sup. $A \neq \bar{O}$. Then$\|A\| \neq 0$.

write $\delta = \frac{\epsilon}{\|A\|}$. Then $\delta > 0$

Let $\bar{x}, \bar{y} \in R^n$ such that $| \bar{x} - \bar{y} | < \delta$.

Consider $|A\bar{x} - A\bar{y}| = |A(\bar{x} - \bar{y})| \leq \|A\| \, | \bar{x} - \bar{y} |$

$$< \|A\|\delta = \|A\| \frac{\epsilon}{\|A\|} = \epsilon.$$

Thus, for every $\epsilon > 0, \forall\delta > 0$

$|\bar{x} - \bar{y}| < \delta \Rightarrow |A\bar{x} - A\bar{y}| <\in \forall\bar{x}, \bar{y} \in \mathrm{R}^n$.

This shows that A is uniformly continuous fromR^ntoR^m.

b) Let $A, B \in L(R^n, R^m)$ and let 'c' be any scalar

For any $\bar{x} \in R^n$with$|\bar{x}| \leq 1$, consider

$|(A + B)\bar{x}| = |A\bar{x} + B\bar{x}| \leq |A\bar{x}| + |B\bar{x}| \leq \|A\||\bar{x}| + \|B\||\bar{x}|$

$= (\|A\| + \|B\|| \bar{x} | \leq \|A\| + \|B\|(\because |\bar{x}| \leq 1)$ ⍰

So, $\{|(\mathrm{A} + \mathrm{B})(\bar{x})| / \bar{x} \in R^n$ with $|\bar{x}| \leq 1\}$ is bounded above by$\|A\| + \|B\|$

$\therefore$ sup. $\{|(A + B)\bar{x}| / \bar{x} \in R^n$With$|\bar{x}| \leq 1\} \leq \|A\| + \|B\|$

$\Rightarrow \|A + B\| \leqslant \|A\| + \|B\|$.

Consider

$$\begin{aligned}\|CA\| &= \sup\{|(cA)(\bar{x})| / \bar{x} \in R^n \text{ with } |\bar{x}| \leq 1\} \\ &= |c|\sup\{|A\bar{x}| / \bar{x} \in R^n \text{ with } |\bar{x}| \leq 1\} \\ &= |c| \, \|A\|.\end{aligned}$$

Define $d: L(R^n, R^m) \to \mathrm{R}$ as $d(A, B) = \|A - B\| \forall A, B \in L(R^n, R^m)$

claims: d is a metric on$L(R^n, R^m)$.

Let$A, B, C \in L\big((R^n, R^m\big)$.

Clearly,

i.$d(A, B) = \|A - B\| \geqslant 0$.

ii.$d(A, B) = 0 \Leftrightarrow \|A - B\| = 0$

$\Leftrightarrow \sup\{|(A - B)(\bar{x})| / \bar{x} \in R^n$ with $|x| \leq 1\} = 0$

$\Leftrightarrow (A - B)(\bar{x}) = 0 \forall\bar{x} \in R^n$ with $|x| \leq 1$

$\Leftrightarrow (A-B)\left(\frac{\bar{x}}{|\bar{x}|}\right) = 0, \forall \bar{x} \in R^n$

$\Leftrightarrow (A-B)(\bar{x}) = 0, \forall \bar{x} \in R^n$

$\Leftrightarrow A(\bar{x}) = B(\bar{x}), \forall \bar{x} \in R^n$

$\Leftrightarrow A = B.$

$\therefore d(A;B) = 0 \Leftrightarrow A = B.$

Consider

iii. $\|A-B\| = \|(-1)(B-A)\| = |-1|\|B-A\| = \|B-A\|$

$\Rightarrow d(A,B) = d(B,A).$

Consider

iv. $\|A-C\| = \|(A-B)+(B-C)\| \leq \|A-B\| + \|B-C\|$

$\Rightarrow d(A,C) \leq d(A,B) + d(B,C).$

$\therefore$ d is a metric on $L(R^n, R^m)$

c.) Let $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$.

Let $\bar{x} \in R^n$ such that $|\bar{x}| \leq 1$.

Consider. $|(BA)(\bar{x})| = \left|B\big(A(\bar{x})\big)\right| \leq \|B\||A\bar{x}|$

$\leq \|B\|\|A\||\bar{x}| \leq \|B\|\|A\|(1)$

So, the set $\{|(BA)(\bar{x})|/\bar{x} \in R^n$ with $|\bar{x}| \leq 1\}$ is bounded above by $\|B\|\|A\|$.

$\therefore \|BA\| \leq \|B\|\|A\|.$

10.3.12 Theorem Let Ω be the set of all invertible linear operators on R^n.

(a) If $A \in \Omega, B \in L(R^n)$, and

$\|B-A\| \cdot \|A^{-1}\| < 1$ then $B \in \Omega$.

(b) Ω is an open subset of $L(R^n)$, and the mapping $A \to A^{-1}$ is continuous on Ω.

(This mapping is also obviously a $1-1$ mapping of Ω onto Ω, which is its own inverse.)

Proof: Let Ω be the set of all invertible linear operators on R^n

(a) Suppose $A \in \Omega, B \in L(R^n)$ and $\|B-A\| \cdot \|A^{-1}\| < 1 \to (1)$

Put $\alpha = \frac{1}{\|A\|}$ and $\beta = \|B-A\|$.

Then by (1), $\beta < \alpha \Rightarrow \alpha - \beta > 0$

For any $\bar{x} \in R^n$, Consider $\alpha|\bar{x}| = \alpha|(\bar{A}^1 A) \cdot \bar{x}|$

$\leq \alpha\|A^{-1}\| \cdot |A\,\bar{x}|$

$= |A\bar{x}| \leq |(A-B)\bar{x}| + |B\bar{x}|$

$$\leq \beta|\bar{x}| + |B\bar{x}| \to (2)$$

Since $\alpha - \beta > 0$, $(\alpha - \beta)|\bar{x}| > 0$ for all $0 \neq \bar{x} \in R^n$.

So, from (2), $|B\bar{x}| \neq 0$ for all$\bar{x} \in R^n \to (3)$

one-one: suppose $x_1, x_2 \in R^n$ such tit $\overline{x_1} \neq \overline{x_2}$.

$$\Rightarrow \bar{x}_1 - \bar{x}_2 \in R^n \text{ and } \bar{x}_1 - \bar{x}_2 \neq 0.$$

So, by (2) ,$B(\overline{x_1} - \bar{x}_2) \neq 0$.

$$\Rightarrow B(\bar{x}_1) \neq B(\bar{x}_2).$$

Onto: the linear operator B on a finite dimensional vector space R^n is one-to-one.

So, by known Result (Th-8.3.5), the range of B is all of R^n.

This shows that B is onto.

Hence, B is a bijective mapping from R^n onto R^n

$\Rightarrow B$ is an invertable linear operator on R^n.

$$\Rightarrow B \in \Omega.$$

Claim:A is an interior point of Ω.

(b) Let $A \in \Omega$. Then A is invertable

$\Rightarrow \|A^{-1}\| \neq 0$.

Write$\delta = \frac{1}{\|A^{-1}\|}$. Then $\delta > 0$.

Consider$S_\delta(A)$; the nbd of A.

Let $B \in S_\delta(A)$

$\Rightarrow \|B - A\| < \delta \Rightarrow \|B - A\| < \dfrac{1}{\|A^{-1}\|} \Rightarrow \|B - A\|\|A^{-1}\| < 1$

So, by part (a), $B \in \Omega$.

$\therefore S_\delta(A) \subseteq \Omega$. Thus Ω is an open setin$L(R^n)$..

Now we show that the mapping

$f: \Omega \to \Omega$, defined by $f(A) = A^{-1} \forall A \in \Omega$, is continuous.

For any$B \in \Omega$, consider $\|B^{-1} - A^{-1}\| = \|B^{-1}AA^{-1} - \bar{B}^{-1}BA^{-1}\|$

$$= \|B^{-1}(A - B)A^{-1}\|$$

$$\leq \|B^{-1}\|\|(A - B)\|\|A^{-1}\| \forall B \in \Omega$$

Since $\|B - A\| \to 0$as $B \to A$, it follows that the R.H.S.of (1) tends to 0 as $B \to A$.

So, from (1),

$\|B^{-1} - A^{-1}\| \to 0asB \to A$.

$\Rightarrow \|f(B) - f(A)\| \to 0$ as $B \to A$.

$\Rightarrow f$ is continuous at A.

$\therefore f$ is continual on Ω.

10.3.13 Matrices

Suppose $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ and $\{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_m\}$ are bases of vector spaces X ad Y, respectively.

Let $A \in L(X, Y)$.

Then $A\bar{x}_j \in X$ for $j = 1$to n.

Since $\{\bar{y}_1, \bar{y}_2, \rightarrow \bar{y}_m\}$ is a boris for Y, $A\bar{x}_j = \sum_{i=1}^{m} a_{ij}\bar{y}_i (1 \leq j \leq n)$.→ (1)

So, for $A \subset L(X, \mathrm{Y})$, we get as set of nos. $a_{ij}, 1 \leq i \leq m \;\; and \;\;\; 1 \leq \delta \leq n$.

We arrange these numbers in a rectangular array of m rows and n Columns, called an m by n matrix:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Observe that the coordinates a_{ij} of the vector $A\bar{x}_j$ (with respect to the bases $\{y_1, y_n\}$appear in the j th column of $[A]$.

The vectors $A\bar{x}_j$, $1 \leq j \leq n$are therefore sometimes called the column vectors of $[A]$.

With this terminology, the range of A is spanned by the column vectors of$[A]$.

Let $\bar{x} \in X$.

Then $\bar{x} = c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n$ for some scalars $c_1, c_2, \dots\dots, c_n$.

So, $\;\; A\bar{x} = A(c_1\bar{x}_1 + c_2\overline{x_2} + \cdots + c_n\bar{x}_n)$

$$\begin{aligned} &= c_1A(\bar{x}_1) + c_2A(\bar{x}_2) + \cdots + c_nA(\bar{x}_n) \\ &= c_1(a_{11}\bar{y}_1 + a_{21}\bar{y}_2 + \cdots + a_{m1}\bar{y}_m) + c_2(a_{12}\bar{y}_1 + a_{22}\bar{y}_2 + \cdots + a_{m2}\bar{y}_m) + \cdots \\ &\cdots + c_n(a_{1n}\bar{y}_1 + a_{2n}\bar{y}_2 + \cdots + a_{mn}\bar{y}_m) \qquad _{\rightarrow}(by(1)) \\ &= (a_{11}c_1 + a_{12}c_2 + \cdots + a_nc_n)\bar{y}_1 + (a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n)\bar{y}_2 + \cdots \\ &\cdots + (a_{m1}a_1 + a_{m2}c_2 + \cdots + a_{mn}c_n)\bar{y}_m. \end{aligned}$$

$$\therefore A\bar{x} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}c_j \right) \bar{y}_i \qquad \rightarrow (2)$$

Thus the Coordinates of $A\bar{x}$ are $\sum_{j=1}^{n} a_{ij}c_j$for $i = 1,2, \dots, m$.

Note that in (1), the summation ranges over the first subscript of a_{ij}, but that we seem over the second subscript when Computing coordinates. Suppose next that an$m$ by n matrix is given, with real entries a_{ij}, so, we have matrix$a_{ij} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$with real entries.

If A is then defined by (2), it is clear that $A \in L(X,Y)$ and that $[A]$ is the given matrix,

$$i.\text{e}; [A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Thus there is a one-to-one correspondence between $L(X,Y)$ and the set of all real m by n matrices.

Observe that $[A]$ depends not only on A but also on the choice of bases of X and Y.

The same A may give rise to many different matrices if we change bases, and vice versa. We shall not pursue this observation any further, since we shall usually work with fixed bases.

If Z is a third vector space, with basis $\{\mathbf{z}_1, \dots, \mathbf{z}_p\}$, if A is given by (1), and if

$$B\mathbf{y}_i = \sum_k b_{ki}\mathbf{z}_k, \quad (BA)\mathbf{x}_j = \sum_k c_{kj}\mathbf{z}_k$$

then $A \in L(X,Y), B \in L(Y,Z), BA \in L(X,Z)$, and since

$$B(A\mathbf{x}_j) = B\sum_i a_{ij}\mathbf{y}_i = \sum_i a_{ij}B\mathbf{y}_i$$

$$= \sum a_{ij}\sum_k b_{ki}\mathbf{z}_k = \sum_k \left(\sum_i b_{ki}a_{ij}\right)\mathbf{z}_k,$$

the independence of $\{\mathbf{z}_1, \dots, \mathbf{z}_p\}$ implies that

$$c_{kj} = \sum_i b_{ki}a_{ij} (1 \le k \le p, 1 \le j \le n) \qquad \rightarrow \quad (3)$$

This shows how to compute the p by n matrix $[BA]$ from $[B]$and $[A]$. If we define the product $[B][A]$ to be $[BA]$, then (3) describes the usual rule of matrix multiplication.

Finally, suppose $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_m\}$ are standard bases of R^n and R^m, and A is given by (4). The Schwarz inequality shows that

$$|A\mathbf{x}|^2 = \sum_i \left(\sum_j a_{ij}c_j\right)^2 \le \sum_i \left(\sum_j a_{ij}^2 \cdot \sum_j c_j^2\right) = \sum_{i,j} a_{ij}^2|\mathbf{x}|^2.$$

Thus

$$\|A\| \le \left\{\sum_{i,j} a_{ij}^2\right\}^{\frac{1}{2}} \qquad \rightarrow \quad (4)$$

If we apply (4) to $B - A$ in place of A, where $A, B \in L(R^n, R^m)$, we see that if the matrix elements a_{ij} are continuous functions of a parameter, then the same is true of A. More precisely:

If S is a metric space, if $a_{11}, \ldots, a_{mn}$ are real continuous functions on S, and if, for each $p \in S, A_p$ is the linear transformation of R^n into R^m whose matrix has entries $a_{ij}(p)$, then the mapping $p \to A_p$ is a continuous mapping of S into $L(R^n, R^m)$.

10.4 SUMMARY:

In this lesson we are discussed about linear transformations of functions with the detailed definitions, examples and theorems.

10.5 TECHNICAL TERMS:

- Fixed point theory
- Inverse function theorem
- Linear operator
- Invertable linear operators

10.6 SELF ASSESSMENT QUESTIONS:

1. If S is a nonempty subset of a vector space X, prove that the span of S is a vector space.
2. Prove that BA is a linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertable.
3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = 0$ only when $\mathbf{x} = 0$. Prove that A is than 1-1.
4. Prove that null spaces and ranges of linear transformations are vector spaces.
5. Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $y \in R^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $|A_1| = |\mathbf{y}|$.

Hint: under certain conditions, quality holds in the Schwarz inequality.

10.7 SUGGESTED READINGS:

1. Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M. Apostal, Narosa Publishing House, 2nd Edition, 1985.

- **Dr. K. Bhanu Lakshmi**

LESSON - 11
DIFFERENTIATION ON LINEAR TRANSFORMATIONS

OBJECTIVES:

After studying the lesson you should able to understand the concept of differentiation on linear transformations.

1. Learn what it means for a function to be differentiable for both single and multivariable functions.
2. Study partial derivatives and how they describe the behavior of functions with more than one variable.
3. Use important differentiation rules, including the Chain Rule, and understand their practical applications.
4. Understand Jacobian matrices and their role in representing the derivatives of functions with multiple variables.
5. Explore the relationship between differentiability and continuity, including the conditions under which functions remain constant.

STRUCTURE:

11.1 INTRODUCTION:

In functional analysis, differentiation of a linear transformation, or Fréchet derivative, is a linear operator that describes the best linear approximation of a function at a point, generalizing the concept of a derivative from single-variable calculus.

11.2 DEFINITIONS AND THEOREMS:

11.2.1 Definitions

(a) Let (a, b) be an open internal in R and f of be a real function defined on (a, b) .We say that f is differentiable at $x \in (a, b)$ if $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ exists and we denote it by $f'(x)$ Thus

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

(b) Let $f_1, f_2, \cdots, f_k$ be real functions defined on a metric space X.

Define a mapping $\bar{f}: x \to R^k$by

$\overline{f}(x) = (f_1(x), f_2(x), \cdots, f_k(x))$for all $x \in X$.

In this case, $f_1, f_2, \ldots, f_k$ are called the components of$\bar{f}$.

Note:

1. $\bar{f}$ is continual on X iff each $f_i$is continuous on X ,for$i = 1$ to k.
2. $\bar{f}$ is differentiable at X iff each f_i is differentiable at X, for $i = 1$ to k.

11.2.2 Definition:

Suppose E is an open set in R^n, $\bar{f}$maps E into R^m, and $\bar{x} \in E$. If there exists a linear transformation A of R^n into R^m such that

$$\lim_{\mathbf{h}\to 0} \frac{|\mathrm{f}(\bar{x}+\bar{h}) - \mathrm{f}(\bar{x}) - A\bar{h}|}{|\bar{h}\ |} = 0 \qquad (1)$$

then we say that $\bar{f}$ is differentiable at $\bar{x}$, and we write

$$\mathrm{f}'(\bar{x}) = A \qquad (2)$$

If $\bar{f}$ is differentiable at every $\bar{x} \in E$, we say that $\bar{f}$ is differentiable in E.

Note:

1) In the above definition$\bar{h} \in R^n$.

If $|\bar{h}|$ is small, then $\bar{x} + \bar{h} \in E$ (since E is open)

Thus $\mathrm{f}(\bar{x}+\bar{h})$ is defined, $\mathrm{f}(\bar{x}+\bar{h}) \in R^m$, and since $A \in L(R^n, R^m), Ah \in R^m$.

Hence $\mathrm{f}(\bar{x}+\bar{h}) - \mathrm{f}(\bar{x}) - A\bar{h} \in R^m$

2) The norm in the numerator of (1) is that of R^m and the norm in the denominator of (1) is that of the R^n.

There is an obvious uniqueness problem which has to be settled before we go any further.

11.2.3 Theorem:

suppose E is an open set in R^n, $\bar{f}$ maps E into R^m, $\vec{x} \in E$, and suppose A_1, A_2 are linear transformations of R^n into R^m such that

$\lim_{\bar{h}\to 0} \frac{|\bar{f}(\bar{x}+\bar{h})-f(\bar{x})-A_1\bar{h}|}{|\bar{h}|} = 0$ and $\lim_{\bar{h}\to 0} \frac{|\bar{f}(\bar{x}+\bar{h})-f(\bar{x})-A_2\bar{h}|}{|\bar{h}|} = 0$ then $A_1 = A_2$.

Proof:

suppose that A_1 ad A_2 are linear transformations of R^n into R^msuch that

$\lim_{\bar{h}\to 0}\frac{|\bar{f}(\bar{x}+\bar{h})-f(\bar{x})-A_1\bar{h}|}{|\bar{h}|}=0$ and $\lim_{\bar{h}\to 0}\frac{|\bar{f}(\bar{x}+\bar{h})-f(\bar{x})-A_2\bar{h}|}{|\bar{h}|}=0 \quad \rightarrow \quad (1)$

Put $B_1 = A_1 - A_2$.

For any $\bar{h}\in R^n$,

Consider

$$\begin{aligned}
&|B\bar{h}| = |A_1\bar{h} - A_2\bar{h}| \\
&= |A_1\bar{h} + f(\bar{x}+\bar{h}) + f(\bar{x}) - f(\bar{x}+\bar{h}) - f(\bar{x}) - A_2\bar{h}| \\
&= |(f(\bar{x}+\bar{h}) - f(\bar{x}) - A_2\bar{h}) - (f(\bar{x}+\bar{h}) - f(\bar{x}) - A_1\bar{h})| \\
&\le |f(\bar{x}+\bar{h}) - f(\bar{x}) - A_1\bar{h}| + |f(\bar{x}+\bar{h}) - f(\bar{x}) - A_2\bar{h}|
\end{aligned}$$

So, $\frac{|B\bar{h}|}{|\bar{h}|}\le\frac{|f(\bar{x}+\bar{h})-f(\bar{x})-A_1\bar{h}|}{|\bar{h}|}+\frac{|f(\bar{x}+\bar{h})-f(\bar{x})-A_2h|}{|\bar{h}|} \quad \forall\, \bar{o}\neq\bar{h}\in R^n$

Taking limit on both sides as$\bar{h}\to 0$, by (1), we get

$$\lim_{h\to 0}\frac{|B\overline{\mathrm{h}}|}{|\bar{h}|}=0.\rightarrow \quad (2)$$

For fixed$\bar{h}\neq 0$, it follows that

$$\frac{|B(t\bar{h})|}{|t\bar{h}|}\rightarrow 0 \text{ as } t\rightarrow 0 \quad \rightarrow (3)(\because \text{ for fixed t} \rightarrow 0 \Rightarrow \mathrm{t\bar{h}} \rightarrow 0).$$

$$\text{Also, } \frac{|B(t\bar{h})|}{|t\bar{h}|}=\frac{|t(B\bar{h})|}{|t\bar{h}|}=\frac{|t||B\bar{h}|}{|t||\bar{h}|}=\frac{|B\bar{h}|}{|\bar{h}|}$$

($\because \beta$ is a linear transformation)

So, the left hand side of (3) is independent of' t '.

$$\begin{aligned}
&\therefore |B\bar{h}| = 0\forall\bar{h}\in\mathbb{R}^n \\
&\Rightarrow B\bar{h} = 0 \Rightarrow (A_1 - A_2)\bar{h} = 0\ \forall\bar{h}\in R^n. \\
&\Rightarrow A_1\bar{h} = A_2\bar{h} \\
&\Rightarrow A_1 = A_2.
\end{aligned}$$

Note

1) The reaction $\lim_{\bar{h}\to 0}\frac{|\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-A\bar{h}|}{|\bar{h}|}=0$ can be written in the form

$$\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})=\bar{f}'(\bar{x})\bar{h}+\gamma(\bar{h})\rightarrow \quad (4)$$

Where $\lim_{\bar{h}\to 0}\frac{|\gamma(\bar{h})|}{|\bar{h}|}=0.$ (In the definition derivative, $A=f'(\bar{x})$)

2) The derivative defined in by (1) or (4) is called the differentiable of $\bar{f}$ at $\bar{x}$ or the total derivative of $\bar{f}$ at $\bar{x}$.

11.2.4 Example

Let $A \in L(R^n, R^m)$. If A is differentiable on $R^n$and$\bar{x} \in R^n$, then $A'(\bar{x}) = A \rightarrow$ (5)

$\left\{\begin{array}{l}\text{Note that } \bar{x} \text{ appears on the left side of (5), but not on the right.} \\ \text{Both sides of are members of } L(R^n, R^m) \text{ where as } \bar{A}x \in R^m.\end{array}\right.$

Proof:

Let $\bar{x} \in R^n$.

Consider $\frac{|\ A(\bar{x}+\bar{h})-A(\bar{x})-A(\bar{h})|}{|\bar{h}|} = \frac{|A(\bar{x})+A(\bar{h})-A(\bar{x})-A(\bar{h})|}{|\bar{h}|}$

(by the linearity of A)

$$= 0$$

$$\Rightarrow \lim_{h \to 0} \frac{|A(\bar{x} + \bar{h}) - A\bar{x} - A\bar{h}|}{|\bar{h}|} = 0$$

$$\Rightarrow A'(\bar{x}) = A.$$

11.2.5 Theorem (Chain Rule):-

suppose E is an open set in R^n, $\bar{f}$ maps E into R^m, $\bar{f}$ is differentiable at $\bar{x}_0 \in E$, $\bar{g}$ maps an open set containing $\bar{f}(E)$ into R^k, and $\bar{g}$ is differentiable at $\bar{f}(x_0)$. Then the mapping F of E into R^k defined by $\bar{\mathrm{F}}(\bar{x}) = \bar{g}\left(\bar{f}(\bar{x})\right)$ is differentiable at $\bar{x}_0$, and $\bar{F}'(x_0) = \bar{g}'\left(\overline{f'}(x_0)\right) \bar{f}'(\bar{x}_0)$

Proof:

Define$\bar{F} : E \rightarrow R^k$ by $\bar{\mathrm{F}}(\bar{x}) = \bar{g}\left(\bar{f}(\bar{x})\right)$ $\forall \bar{x} \in \mathrm{R}$

Claim: F is differentiable at $\bar{x}_0 \in E$and $\bar{F}'(\bar{x}_0) = \bar{g}'\left(\bar{f}(\bar{x}_0)\right) f'(\bar{x}_{0\,)})$.

Let $\bar{y}_0 = f(\bar{x}_0)$.

Given $\bar{f}$ is differentiable at $\bar{x}_o$ and $\bar{g}$ is differentiable at $\bar{y}_0$.

So , $\bar{f}'(\bar{x}_0)$ ad $\bar{g}'(\bar{y}_0)$ exists.

Put $A = \bar{f}'(\bar{x}_0)$add $B = \bar{g}'(\bar{y}_0)$.

Then A is a linear transformation from R^n into R^m, B is a linear transformation from R^m into R^k, and

$$\lim_{\bar{h} \to 0} \frac{|\bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0) - A\bar{h}|}{|\bar{h}|} = \bar{0} \quad \rightarrow (1)$$

$$\lim_{\bar{h} \to 0} \frac{|\bar{g}(\bar{y}_0 + \bar{k}) - \bar{g}(\bar{y}_0) - B\bar{k}|}{|\bar{k}|} = \bar{0} \quad \rightarrow (2)$$

Define

$$\left.\begin{array}{ll} \bar{u}(\bar{h}) = \bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0) - A\bar{h} & \forall \bar{h} \in R^n \text{ ad} \\ \bar{v}(\bar{k}) = \bar{g}(\bar{y}_0 + \bar{k}) - \bar{g}(\bar{y}_0) - B\bar{k} & \forall \bar{k} \in \mathrm{R}^m \end{array}\right\}. \quad \rightarrow (3)$$

So

(1) $\Rightarrow \lim_{\bar{h}\to 0} \frac{|\bar{u}(\bar{h})|}{|\bar{h}|} = \bar{0}$ and (2) $\Rightarrow \lim_{\vec{k}\to 0} \frac{|\bar{v}(R)|}{|\bar{k}|} = 0$

$$\left.\begin{array}{l} \therefore \dfrac{|\bar{u}(\bar{h})|}{|\bar{h}|} = \epsilon(\bar{h}) \text{ where } \in(\bar{h}) \to 0 \text{ as } \bar{h} \to 0 \text{ and} \\ \dfrac{|\bar{v}(\bar{k})|}{|\bar{k}|} = \eta(\bar{k}) \text{ where } \eta(\bar{k}) \to 0 \text{ as } k \to 0 \end{array}\right\} \to (4)$$

For given $\bar{h}$, put $\bar{k} = \bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0)$. → (5)

$$\text{Now } |\bar{k}| = |\bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0)| = |\bar{u}\cdot(\bar{h}) + A\bar{h}| \quad \text{(from (3))}$$
$$\leq |\bar{u}(\bar{h})| + |A\bar{h}| \leq |\bar{u}(\bar{h})| + ||A|||\bar{h}|$$
$$= \epsilon(\bar{h})|\hbar| + \|A\||\bar{h}|(\text{from}(4)).$$

$$\therefore |\bar{K}| \leqslant (\epsilon(\bar{h}) + \|A\|)([\bar{h}\rangle) \quad \to (6)$$

Consider $\frac{|F(\bar{x}_0+\bar{h}) - F(\bar{x}_0) - BA\bar{h}|}{|\bar{h}|} =$

$$= \frac{1}{|\bar{h}|}\left\{\left|\bar{g}\left(\bar{f}(\bar{x}_0 + \bar{h})\right) - \bar{g}\left(\bar{f}(\bar{x}_0)\right) - BA\bar{h}\right|\right\}\left(\because F = g(f(\bar{x}))\right).$$

$$= \frac{1}{|\bar{h}|}|\bar{g}(\overline{y_0} + \bar{k}) - \bar{g}(\bar{y}_0) - BA\bar{h}| \quad \because f(\overline{x_0}) = y_0$$

$$(5) \Rightarrow \bar{f}(\bar{x}_0 + \bar{h}) - \bar{f}(\bar{x}_0) = \bar{k}$$

$$= \frac{1}{|\bar{h}|}|\bar{\vartheta}(\bar{k}) + B\bar{k} - BA\bar{h}|(\text{from (3)}$$

$$= \frac{1}{|\bar{h}|}|\bar{\vartheta}(\bar{r}) + B(\bar{K} - A\bar{h})|$$

$$= \frac{1}{|\bar{h}|}\left|\bar{\vartheta}(\bar{k}) + B\left(\bar{u}(\bar{h})\right)\right| \text{(from (3) \& (5))}$$

$$\leq \frac{1}{|\bar{h}|}|\bar{\vartheta}(\bar{k})| + \frac{\left||\vec{B}|\right| |\bar{u}(\bar{h})|}{|\bar{h}|}$$

$$= \frac{\eta(\bar{k})|\bar{k}|}{|\bar{h}|} + \|B\|\frac{|\bar{u}(\bar{h})|}{|\bar{h}|} \cdot \text{(from (4))}$$

$$\leq \eta(\bar{k})(\epsilon(\bar{h}) + \|A\|) + \|B\|\frac{|\bar{u}(\bar{h})|}{|\bar{h}|} \cdot (\text{from}(6)).$$

$$\therefore \frac{|\bar{F}(\bar{x}_0 + \bar{h}) - \bar{F}(\bar{x}_0) - BA\bar{h}|}{|\bar{h}|} \leq \eta(\bar{k})(\epsilon(\bar{h}) + \|A\|) + \|B\|\frac{|\bar{u}(\bar{h})|}{|\bar{h}|}$$

Taking limits on both sides as $\bar{h} \to 0$, by (6),

we have $\eta(\bar{k}) \to 0$ and $\frac{|\bar{u}(\bar{h})|}{|\bar{h}|} \to 0$.

$$\therefore \lim_{h \to 0} \frac{\left|\bar{F}(\bar{x}_0 + \bar{h}) - \bar{F}(\bar{x}_0) - BA\bar{h}\right|}{|\bar{h}|} = 0.$$

$$\Rightarrow \bar{F}'(\bar{x}_0) = BA = \bar{g}'(\bar{y}_0)\bar{f}'(\bar{x}_0) = \bar{g}'\left(\bar{f}(\bar{x}_0)\right)\bar{f}'(\bar{x}_0)$$

$$\therefore \bar{F}'(\overline{x_0}) = \bar{g}'\left(\bar{f}(\overline{x_0})\right)\bar{f}'(\bar{x}_0).$$

11.3 PARTIAL DERIVATIVES:

Let E be an open set in R^n, and let $\{\bar{e}_1, \bar{e}_2, \to \bar{e}_n\}$ and $\{\bar{u}_1, \bar{u}_2, -\bar{u}_m\}$ be the standard bases of R^n ad R^m respectively. Suppose $\bar{f}$ maps Einto R^m : The components of $\bar{f}$ are the real functions $f_1, f_2, \dots, f_m$ defined by $\bar{f}(\bar{x}) = \left(f_1(\bar{x}), f_2(\bar{x}), \dots\dots, f_n(\bar{x})\right)$

$$= \sum_{i=1}^{m} f_i(\bar{x})\mathrm{u}_{\boldsymbol{i}}(\mathbf{x} \in E)$$

or, equivalently, by $f_i(\bar{x}) = \mathrm{f}(\bar{x}) \cdot \mathrm{u}_i, 1 \leq i \leq m$

For $\bar{x} \in E$ ad for $1 \leq i \leq m, i \leq j \leq n)$

we define $\left(D_j f_i\right)(\bar{x}) = \lim_{t \to 0} \frac{f_i\left(\bar{x} + (\mathrm{t}\bar{e}_j\right) - f_i(\bar{x})}{t}$, provided the limit exists.

Writing$(x_1, x_2, \dots, x_n)$inplace of $\bar{x} \in E$, We see that $D_j f_i$ is the deravatave &f_i with respect x_j keeping $(x_1, x_2, \to x_n)$ in place of $\bar{x} \in E$, we see that $D_j f_i$ is the devalative of f_i with respect to x_j, keeping the other variables fixed. It is denoted by $\frac{\partial f_i}{\partial x_j}$. Thus

$\left(D_j f_i\right)(x_1, \dots, x_n) = \frac{\partial f_i}{\partial x_i}(x_1, \dots, x_n)$for $\frac{\partial f_i}{\partial x_j} = (x_1, \dots, x_n) \in E$.

Here $D_j f_i$ is called as partial derivative.

In many cases where the existence of a derivative is sufficient when dealing with functions of one variable, continuity or at least boundedness of the partial derivatives is needed for functions of several variables.

11.3.1 Theorem: Suppose $\bar{f}$ maps an open set $E \subseteq R^n$into R^m, and $\bar{f}$ is differentiable at a point $\bar{x} \in E$. Then the partial derivatives $\left(D_j f_i\right)(\bar{x})$ exist, and $\bar{f}'(\bar{x})\bar{e}_j = \sum_{i=1}^{m} \left(D_j f_i\right)(\bar{x})\overline{u_i}(1 \leq j \leq n)$.

Proof:

Let $E \subseteq R^n$.

Suppose $\bar{f}$ maps E into R^m and $\bar{f}$ is differentiable at $\bar{x} \in E$.

Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ and $\{\bar{u}_1, \rightarrow \overline{u_m}\}$ be the standard bases of $R^n$and R^m, respectively.

Take $\in > 0$.

Since $\bar{f}$ is differentiable at $\bar{x}$, $\lim_{\bar{h}\to 0} \frac{|\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-f'(\bar{x})\bar{h}|}{|\bar{h}|} = 0$

ie, $\exists\delta > 0 \cdot \frac{|\bar{f}(\bar{x}+\bar{h})-\bar{f}(\bar{x})-f'(\bar{x})\bar{h}|}{|\bar{h}|} < \epsilon$ when ever $0 < |\bar{h}| < \delta$

If $t \in R$ such that $0 < |t| < \delta$, then $|t\bar{e}_j| = |t| < \delta$.

So, from (1), $\frac{|\bar{f}(\bar{x}+t\bar{e}_j)-\bar{f}(\bar{x})-\bar{f}'(\bar{x})t\bar{e}_j|}{|t\bar{e}_j|} < \epsilon$ whenever $|t\bar{e}_j| = |t| < \delta$.

$\Rightarrow \left|\frac{\bar{f}(\bar{x}+t\bar{e}_j)-\bar{f}(\bar{x})}{t} - \frac{t\,\bar{f}'(\bar{x})\bar{e}_j}{t}\right| < \epsilon$ Whenever$|t\bar{e}_j| = |t| < \delta$.

$$\Rightarrow \lim_{t\to 0} \frac{|\bar{f}(\bar{x}+t\bar{e}_j) - \bar{f}(\bar{x})|}{|t|} = f'(\bar{x})e_j \qquad \rightarrow (1)$$

we have $\bar{f}(\bar{x})= \left(f_1(\bar{x}), f_2(\bar{x}), \ldots, f_m(\bar{x})\right)$

$$= \sum_{i=1}^{m} f_i(\bar{x})\bar{u}_i \forall f_{\text{for all}}\, \bar{x} \in E.$$

So, (1) $\Rightarrow \bar{f}'(\bar{x})\bar{e}_j = \lim_{t\to 0} \sum_{i=1}^{m} \frac{f_i(\bar{x}+t\bar{e}_j) - f_i(\bar{x})}{t} \bar{u}_i \in (\bar{x})$

ie, $\bar{f}'(\bar{x})\bar{e}_j = \sum_{i=1}^{m} (D_j f_i)(\bar{x})\bar{u}_i$ for all $j = 1$ to n.

Note: Consequences of Theorem

We know that there is a one-to-one correspondence between matrices and linear transformations.

Let $[\bar{f}'(\bar{x})]$ be the matrix the represents $\bar{f}'(\bar{x})$ with respect to the standard bases $\{\bar{e}_1, \ldots, \bar{e}_n\}$ of R^n and $\{\bar{u}_1, \ldots, \overline{u_m}\}$of R^m. Then $\bar{f}'(\bar{x})\bar{e}_j$ is the j^{th} column $[\bar{f}'(\bar{x})]$ and

$$[\bar{f}'(\bar{x})] = \begin{bmatrix} (D_1f_1)(\bar{x}) & (D_2f_1)(\bar{x}) & \cdots & (D_nf_1)(\bar{x}) \\ (D_1f_2)(\bar{x}) & (D_2f_2)(\bar{x}) & \cdots & (D_nf_2)(\bar{x}) \\ \vdots & \vdots & & \vdots \\ (D_1f_m)(\bar{x}) & (D_2f_m)(\bar{x}) & \cdots & (D_nf_m)(\bar{x}) \end{bmatrix}$$

This matrix is called the Jacobin matrix of $\bar{f}'$at$\bar{x}$. Sometimes we denote it as $(D\bar{f})(\bar{x})$.

11.3.2 Example: Let$(a, b) \subset R'$, $E \subset R^n$ be open sets and let $\gamma: (a, b) \rightarrow E$ be a differentiable curve. ($\gamma$is continuous and γ' is exists and is continuous in (a,b)) let f be a real-valued differentiable function with domain$E$. i.e, $f: E \to R'$.

Define $g(t) = f(t) = (f \circ \gamma)(t) = f(f(t)), (a < (t < b)$ →(1)

By chain Rule, $g'(t) = f'(\gamma(t))g'(t), a < t < b$ → (2)

Since $\gamma'(t) \in L(R, R^n)$ and $f'(\gamma(t)) \in L(R, R^n)$,

by (2), $g'(t) \in L(R, R)$.

ie, $g'(t)$ is a linear operator on R

This agrees with the fact that g maps (a, b) into R

However, $g'(t)$ can also be identified as a real number.

(Now we compute the no. $g'(t)$in terms of the partial derivatives of f and the derivatives of the components of γ)

Let $\{\bar{e}_1, \dots, \bar{e}_n\}$ be the standard basis of R^n and

let $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)) \in R^n$.

Now $\gamma'(t) = (\gamma_1'(t), \dots, \gamma_n'(t)) = \sum_{i=1}^{n} \gamma_{\mathrm{i}}'(t)\overline{e_i}$ → (3)

So, $[\gamma'(t)]$ is the $n \times 1$ matrix which has $\gamma_{\mathrm{i}}'(t)$ in the i^{th} row.

$$\text{ie, } [\gamma'(t)] = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}_{n\times 1}$$

For every $\bar{x} \in E, [f'(\bar{x})]$ is the $1 \times n$ matrix which has $(D_j f)(\bar{x})$ in the j^{th} column,

$$\text{i.e}[f'(\bar{x})] = [(D_1 f)(\bar{x}) \cdot (D_2 f)(\bar{x}) \cdots (\mathrm{D}_n f)(\bar{x})]_{1\times n}$$

$$\therefore \quad [g'(t)] = [f'(\gamma(t))][\gamma'(t)] = [(D_1 f)(\gamma(t)) - (D_2 f)\gamma(t) \cdots (D_n f)\gamma(t)]\begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$$

$[g'(t)] = \sum_{i=1}^{n} (D_i f)\gamma(t)\gamma'(t)$ → (4)

Def For each$\bar{x} \in E$, the gradient of f at $\bar{x}$is defined as

$$(\nabla f)(\bar{x}) = \sum_{i=1}^{n} (D_i f)(\bar{x})\overline{e_i}$$

From (3) & (4) $g'(t) = \sum_{i=1}^{n} (D_i f)(\gamma(t))\gamma'(t)$

$$=[(D_1 f)(\gamma(t)) - (D_2 f)\gamma(t) \cdots (D_n f)\gamma(t)]\begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$$

$$= (\nabla f)\gamma(t)\sum_{i=1}^{n} \gamma'(t)\overline{e_i} = (\nabla f)\gamma(t)\gamma'(t)$$

Fix $\bar{x} \in E$. Let $\bar{u} \in R^n$ be a unit vector →(5)

i. e, $|\bar{u}| = 1$

Put $\gamma(t) = \bar{x} + t\bar{u}(-\infty < t < \infty)$. Then $\gamma'(t) = \bar{u}, \forall t$.

So, from (4), $g'(0) = (\nabla f)\gamma(0)\gamma'(0) = (\nabla f)(\bar{x})\bar{u}$ $\rightarrow (6)$

$$\therefore \lim_{t\to 0} \frac{f(\bar{x} + t\bar{u}) - f(\bar{x})}{t} = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = (\nabla f)(\bar{x})\bar{u} \quad \rightarrow (7)$$

The limit in(7) is usually the directional derivative of f at $\bar{x}$, in the direction of the unit vector $\bar{u}$, and is denoted by $(D_{\bar{u}}f)(\bar{x})$.

$$\text{Thus } (D_{\bar{u}}f)(\bar{x}) = \lim_{t\to 0} \frac{f(\bar{x} + t\bar{u}) - f(\bar{x})}{t} = (\nabla f)(\bar{x})\bar{u} \quad \rightarrow (8)$$

If $\bar{u} = \sum_{i=1}^{n} u_i\bar{e}_i$, then from (8), $(D_{\bar{u}}f)(\bar{x}) =$

$$= \big((D_1 f)(\bar{x}), (D_2 f)(\bar{x}), \rightarrow (D_n f)(x)\big)(u_1, u_2, \cdots, u_n) = \textstyle\sum_{i=1}^{n} (D_i f)(\bar{x})u_i \quad \rightarrow (9)$$

ie, $(D_{\bar{u}}f)(\bar{x})$ can be expressed in terms of the partial derivatives of f at $\bar{x}$.

In particular, if $\dot{\bar{u}} = \overline{e_i}$, then (9) becomes $\big(D_{\overline{e_i}}f\big)(\bar{x}) = (D_i f)(\bar{x})$ which is the partial derivative of f w.r.to $\bar{x}$.

11.3.3 Definition: A subset E of R^n is called a convex set if, for any $\bar{x}, \bar{y} \in E$

$$0 \le \lambda \le 1, \lambda\bar{x} + (1 - \lambda)\bar{y} \in E.$$

11.3.4 Theorem

Suppose $\bar{f}$ maps a convex open set $E \subset R^n$ into R^m, $\bar{f}$ is differentiable in E, and there is a real number M such that $\|\bar{f}'(x)\| \le M$ for every $\bar{x} \in E$. Then $|\bar{f}(\bar{b}) - \bar{f}(\bar{a})| \le M|\bar{b} - \bar{a}|$ for all $\bar{a} \in E, \bar{b} \in E$.

Proof: Let $E \in R^n$ and E is a convex set in R^n.

suppose$\bar{f}$ maps E into $R^m, \bar{f}$ is differentiable in E and there is a real neember$M$ such that $\|\bar{f}'(x)\| \le M$ for all $\bar{x} \in$ E.

Fix $\bar{a}, \bar{b} \in E$. Define $\gamma: [0,1] \rightarrow E$ as

$\gamma(t) = (1 - t)\bar{a} + t\bar{b}$ for all $t \in [0,1]$.

Now $\gamma(1) = \bar{b}$& $\gamma(0) = \bar{a}$.

Since E is convex, $\gamma(t) \in E$ for all $t \in [0,1]$.

So, γ is differentiable on [0,1] and $\gamma'(t) = (\bar{0} - \bar{1})\bar{a} + \bar{b} = -\bar{a} + \bar{b}$ $\rightarrow (1)$

put $\bar{g}(t) = \bar{f}(\gamma(t))$ for all $t \in [0,1]$.

Then $\bar{g}$ is differentiable on [0,1], $\bar{g}(1) = \bar{f}(f(1)) = \bar{f}(\bar{b}), \bar{g}(0) = \bar{f}(\gamma(0)) = \bar{f}(\bar{a})$,

and

$g'(t) = \bar{f}'(\gamma(t))\gamma'(t)$

$= \bar{f}'(\gamma(t))(\bar{b} - \bar{a})$

$\Rightarrow |\bar{g}'(t)| = |\bar{f}'(\gamma(t))(\bar{b} - \bar{a})|$

$\leq \|\bar{f}'(\gamma(t))\| \, |\bar{b} - \bar{a}|$

$\leq M|\bar{b} - \bar{a}| \rightarrow (2)$

(By Chain Rule)

$\Rightarrow |\bar{g}'(t)| \leq M|\bar{b} - \bar{a}| \forall t \in [0,1].$

Since $\bar{g}: [0,1] \rightarrow R^n$ is continuous and differentiable on $[0,1]$,

Then $|\bar{g}(1) - \bar{g}(0)| \leq |\bar{g}'(\mathrm{t})|(1 - 0)$ for some $t \in [0,1]$.

$\Rightarrow |\bar{f}(b) - \bar{f}(a)| \leq |\bar{g}'(\mathrm{t})| \quad \rightarrow (3)(\bar{g}(1) = \bar{f}(b) \& \bar{g}(0)) = \bar{f}(a))$

From (2) & (3), $|\bar{f}(b) - \bar{f}(a)| \leq M \mid \bar{b} - \bar{a} \mid$ for all $\bar{a}, \bar{b} \in E$.

2.) If $\bar{h} = \sum_{j=1}^{n} h_j \bar{e}_j$, then $f'(\bar{x})\bar{h} = f'(\bar{x})(\sum_{j=1}^{n} h_j \bar{e}_j)$

$$= \sum_{j=1}^{n} h_j f'(\bar{x})\bar{e}_j \quad (\because f'(\bar{x}) \text{ is linear }).$$

$$= \sum_{j=1}^{n} h_j \left(\sum_{i=1}^{m} (D_j f_i)(\bar{x})\bar{u}_i \right)$$

11.3.5 Corollary

Suppose $\bar{f}$ maps a convex open set $E \subseteq R^n$ into R^m, $\bar{f}$ is differentiable in E and $\bar{f}'(x) = 0$ for all $x \in E$. Then $\bar{f}$ is constant.

Proof:

Suppose $\|\bar{f}'(x)\| \leq 0$ for all $x \in E$.

Then by the above Theorem, $|\bar{f}(\bar{b}) - \bar{f}(\bar{a})| \leq (0)|\bar{b} - \bar{a}|$ for all $\bar{a}, \vec{b} \in E$

$\Rightarrow \bar{f}(\bar{b}) - \bar{f}(\bar{a}) = \bar{0}$ for all $x \in E$.

$\Rightarrow \bar{f}(\bar{a}) = \bar{f}(\bar{b})$ for all $x \in E$.

This shows that $\bar{f}$ is a constant function on E.

11.3.6 Definition

A differentiable mapping $\bar{f}$ of an open set $E \subseteq R^n$ into R^m is said to be continuously differentiable in E if f' is a continuous mapping of E into $L(R^n, R^m)$.

(ie., for each $\bar{x} \in E$ and for each $\in > 0, \exists \delta > 0)$

$\|\bar{f}'(\bar{y}) - \bar{f}'(\bar{x})\| < \epsilon$ whenever $\bar{y} \in E$ with $|\bar{x} - \bar{y}| < \delta$.,

Note: In this above definition we also say that $\bar{f}$ is a C'-mapping, or $\bar{f} \in C'(E)$

11.3.7 Theorem

Suppose $\bar{f}$ maps an open set $E \subset R^n$ into R^m. Then $\bar{f} \in C'(E)$ if and only if the partial derivatives D, f_i exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

Proof: Given that $\bar{f}$ maps an open set $E \subseteq R^n$into R^m.

Assume that$\bar{f} \in C'(E)$.

Then $\bar{f}'$ is continuous in E.

Let $\bar{x} \in E$ and$\in > 0$. Let $\{\bar{e}_1, \dots, \bar{e}_n\}$ ad $\{\bar{u}_1, \dots, \bar{u}_m\}$ be the standard bases of R^n and R^m, respectively

since $\bar{f}'$ is continuous at $\bar{x}$, for $\in > 0, \exists \delta > 0 \Rightarrow| \{\bar{f}'(\bar{y}) - \bar{f}'(\bar{x})|| < 0$ whenever $\bar{y} \in E$ with $|\bar{X} - \bar{y}| < \delta$. Fix $\bar{y} \in E$ with$|\bar{x} - \bar{y}| < \delta$.

By the definition of norm, $\|\bar{f}'(\bar{y}) - \bar{f}'(\bar{x})\| =$

$$= \sup\left\{\left|\left(\bar{f}'(\bar{y}) - \bar{f}'(\bar{x})\right)(\bar{z})\right| / \bar{z} \in R^n \text{ and } |\bar{z}| \le 1\right\} \quad \rightarrow (1)$$

Since$|\bar{e}_j| = 1$ then $|\bar{f}'(\bar{y})\bar{e}_j - \bar{f}'(\bar{x})\bar{e}_j| \le \|\bar{f}'(\bar{y}) - \bar{f}'(\bar{x})\| < \in$ for $j = 1 to\ n \rightarrow (2)$

Since $\bar{F}$ is differentiable on E, by theorem (4.17), the partial derivatives

$$(D_j f_i)(\bar{x}) \text{ exists and } \bar{f}'(\bar{x})\bar{e}_j = \sum_{i=1}^{m} (D_j f_i)(\bar{x})\bar{u}_i \forall \bar{x} \in E$$

$$\Rightarrow \bar{f}'(\bar{x})\bar{e}_j\bar{u}_i = (D_j f_i)(\bar{x}) \forall \bar{x} \in E \text{ and } \forall i,j \text{ with } 1 \le i \le m, 1 \le j \le n.$$

Consider $(D_j f_i)\bar{y} - (D_j f_i)\vec{x}) = \bar{f}'(\bar{y})\overline{e_j}\bar{u}_i - \bar{f}'(\bar{x})\bar{e}_j\bar{u}_i = (f'(\bar{y}) - f'(\bar{x}))(e_j)\overline{u_i}$

$$\Rightarrow |(D_j f_i)(\bar{y}) - (D_j f_i)(\bar{x})| = |(\bar{f}'(\bar{y})\bar{e}_j - f'(\bar{x})\overline{e_j}) \cdot \bar{u}_i|$$

$\le |(\bar{f}'(\bar{y})\bar{e}_j - f'(\bar{x})\overline{e_j})|$ (Since$|\bar{u}_i| = 1$)

$$\le \|\bar{f}'(\bar{y}) - \bar{f}'(\bar{x})\| < \in$$

$\bar{y} \in E$with $|\bar{y} - \bar{x}| < \delta$ $(by\ (2))$.

Thus for $\epsilon > 0, \exists\ a\ \delta > 0\ \partial |(D_j f_i)(\bar{y}) - (D_j f_i)(\bar{x})| < \in$

When ever $\bar{y} \in E$ and$|\bar{y} - \bar{x}| < \delta$.

$\Longrightarrow D_j f_i$is continuous at $\bar{x}$ ∀ i and j.

This is true for every $\bar{x} \in E$.

$\therefore D_j f_i$is continuous on E for all $1 \le i \le$ m and $1 \le j \le n$.

Conversely, suppose that the partial derivatives $D_j f_i$ exist and are continuous on $E$for $1 \le i \le m, 1 \le j \le n$, where $\bar{f} = (f_1, f_2, \dots, f_m)$.

We have each f_i is a real function on E.

By a Known Result,$\bar{f}'$ is continuous on E iff each f_i' is Continuous on E.

ie,$\bar{f}$ is a $\mathcal{C}'$-mapping iff each f_i is a $\mathcal{C}' -$ mapping.

So, it is enough if we show that each f_i is a $\mathcal{C}' -$ mapping.

Let $\bar{x} \in E$ and $\in > 0$ We may assume that $f_i = f$ for fixed i with $1 \le i \le m$.

Since $\bar{x} \in E$ and E is open, $\exists\ \delta_c > 0 \Rightarrow S_{\delta_c}(\bar{x}) \subseteq E$.

Since $D_j f$ is continucus at $\bar{x}, \exists \delta_j > 0\ \exists \left|(D_j f)(\bar{x}) - (D_j f)(\bar{y})\right| < \frac{\epsilon}{n}$ whenever $\bar{y} \in E$ and

$|\bar{x} - \bar{y}| < for 1 \le j \le n$

write$\delta = \min\{\delta_0, \delta_1, \dots, \delta_n\}$.

Then $\delta > 0, S_\delta(\bar{x}) \subseteq E$ and $\left|(D_j f)(\bar{x}) - (D_j f)(\bar{y})\right| < \frac{\epsilon}{n}$ whenever

$\bar{y} \in E$ with $|\bar{x} - \bar{y}| < \delta$ for $1 \le j \le n$. $\rightarrow$ (3)

Claim: $f'(\bar{x})\bar{e}_j = \sum_{j=1}^{n} D_j f(\bar{x})$.

Suppose $\bar{h} = (h_1, h_2, \dots, h_n) = \sum_{j=1}^{n} h_j \bar{e}_j$ and $|\bar{h}| < \delta$.

put $\bar{\vartheta}_0 = \overline{0}$ and $\bar{\vartheta}_k = h_1\bar{e}_1 + h_2\bar{e}_2 + \cdots + h_k\bar{e}_k$ for $1 \leqslant k \leqslant n$.

Then $\bar{\vartheta}_k = h_1\bar{e}_1 + h \cdots + h_{k-1}\bar{e}_{k-1} + h_k\bar{e}_k = \bar{\vartheta}_{k-1} + h_k\bar{e}_k$ for $k = 1$ to n.

It follows that

So $f(\bar{x} + \bar{h}) - f(\bar{x}) = \sum_{j=1}^{n} \left[f(\bar{x} + \bar{\vartheta}_j) - f(\bar{x} + \bar{\vartheta}_{j-1})\right]$

Since each $|\bar{\vartheta}_j| < \delta, 1 \le j \le n$ and $S_{\delta_0}(\bar{x})$is convex, we have

the segments with end points $\bar{x} + \bar{\vartheta}_{j-1}, \bar{x} + \bar{\bar{\vartheta}}_j$ lie in $S_{\delta_0}(\bar{x})$

Define $\phi: [0,1] \to R$ as $\phi(t) = f(\bar{x} + \bar{\vartheta}_{j-1} + t h_j \bar{e}_j)$ for all $t \in [0,1]$.

Now $\quad \phi(1) = f(\bar{x} + \bar{\vartheta}_{j-1} + h_j\bar{e}_j) = f(\bar{x} + \bar{\vartheta}_j)$ and

$\phi(0) = f(\bar{x} + \bar{\vartheta}_{j-1})$.

Since. $D_k f$ exits in E, we get ϕ is differentiable on [0,1] and

$$\phi'(t) = \frac{d}{dt}(\phi(t)) = \frac{d}{dt}\left(f(\bar{x} + \bar{\vartheta}_{j-1} + t h_{j'}\bar{e}_j)\right):$$

$$= h_j D_j f(\bar{x} + \bar{\vartheta}_{j-1} + t h_{j'}\bar{e}_j)$$

By Mean value Theorem, $\exists \theta_j$ in $(0,1)$ such that $\phi(1) - \phi(0) = \phi'(\theta_j)(1 - 0)$.

$$\Rightarrow f(\bar{x} + \bar{\vartheta}_j) - f(\bar{x} + \bar{\vartheta}_{j-1}) = h_j D_j f(\bar{x} + \bar{\vartheta}_{j-1} + \theta_j h_j \bar{e}_j)$$

$$\therefore f(\bar{x} + \bar{h}) - f(\bar{x}) = \sum_{j=1}^{n} \left[f(\bar{x} + \bar{\vartheta}_j) - f(\bar{x} + \bar{\vartheta}_{j-1})\right]$$

$$= \sum_{j=1}^{n} h_j D_j f(\bar{x} + \bar{\vartheta}_{j-1} + \theta_j h_j \overline{e_j})$$

$$\Rightarrow f(\bar{x}+\bar{h})-f(\bar{x})-\sum_{j=1}^{n} n_j(D_jf)(\bar{x})=\sum_{j=1}^{n} h_jD_jf(\bar{\vartheta}_{j-1} \ +\theta_jh_j\overline{e_j})$$

$$\Rightarrow \left|f(\bar{x}+\bar{h})-f(\bar{x})-\sum_{j=1}^{n} n_j(D_jf)(\bar{x})\right|=\left|\sum_{j=1}^{n} h_jD_jf(\bar{\vartheta}_{j-1} \ +\theta_jh_j\overline{e_j})\right|$$

$$\leq \sum_{j=1}^{n} |h_j||(D_jf)(\bar{x}+\bar{J}_{j-1}+\theta_jh_j\bar{e}_j)-(D_jf)(\bar{x})|\leq \sum_{j=1}^{n} |h_j|\frac{\epsilon}{n}$$

(by (3))

$$\Rightarrow |f(\bar{x}+\bar{h})-f(\bar{x})-\textstyle\sum_{j=1}^{n} n_j(D_jf)(\bar{x})|<\epsilon|\bar{h}| \text{for all } \bar{h} \text{ such that } |\bar{h}|<\delta.$$

$$\Longrightarrow \frac{|f(\bar{x}+\bar{h})-f(\bar{x})-\sum_{j=1}^{n} n_j(D_jf)(\bar{x})|}{|\bar{h}|}<\epsilon \text{ When ever } |\bar{h}|<\delta$$

$\therefore f$ is differentiable at $\bar{x}$

and that $f'(\bar{x})$ is the linear function which assigns the number $\sum h_j(D_jf_j)(\bar{x})$ to the vector $\bar{h}=\sum h_j\bar{e}_j$, ie. if $\bar{h}=\sum_{j=1}^{n} h_j\bar{e}_j$, then $f'(\bar{x})\bar{h}=\sum_{j=1}^{n} h_j(D_jf)(\bar{x})$.

The matrix $[f'(\bar{x})]$ consists of the row $[(D_1f)(\bar{x})(D_2f)(\bar{x}) \ldots\ldots\ldots.(D_nf)(\bar{x})]$

Since $D_1f, D_2f, \ldots D_nf$ are continuous functions on E, it follows that $f'$is continuous on E, $i.e\ \bar{f}\in\mathcal{C}'(E)$.

11.4 SUMMARY:

In this lesson we are discussed about differentiation of functions with the detailed definitions, examples and theorems.

11.5 TECHNICAL TERMS:

- Chain rule
- Partial derivatives
- Jacobian matrix
- Convex set
- Continuously differentiable

11.6 SELF ASSESSMENT QUESTIONS:

1. If $f(0,0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2} \text{ if } (x, y) \neq (0,0)$$

Prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of R^2, although f is not continuous at (0,0).

2. Suppose that f is a real valued function defined in an open set $E \subset R^n$ and that the partial derivatives $D_1 f .... D_{n1} f$ are bounded in E .Prove that f is continuous in E

3. Suppose that f is a real valued function defined in an open set $E \subset R^n$ and that f has a local maximum at a point $x \in E$ prove that $f'(x) = 0.$

4. If f is differentiable mapping of a connected open set $E \subset R^n$ in to R^m ,and if $f'(x) = 0$ for every $x \in E$, prove that f is constant in $E.$

5. If f and g are differentiable real functions in R^n, prove that

$\nabla(fg) = f\nabla g + g\nabla f$ and that $\nabla(1/f) = -f^{-2}\nabla f$ where $f \neq 0$

6. Explain the role of the Jacobian matrix in the context of differentiable mappings. How does the Jacobian relate to the partial derivatives of the function?

11.7 SUGGESTED READINGS:

1. Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2ndEdition,1985.

- **Dr. K. Bhanu Lakshmi**

LESSON- 12

CONTRACTION MAPPINGS AND THE INVERSE FUNCTION THEOREM

OBJECTIVES:

The objectives of this study are to explore key mathematical concepts related to metric spaces, integrals, and differentiability. These include:

1. To apply the contraction mapping theorem in complete metric spaces, demonstrating the existence of unique fixed points.
2. To analyze the relationship between differentiation and integrals, providing essential knowledge for advanced calculus.
3. To understand the conditions for local invertibility using the inverse function theorem and its implications.
4. To establish the continuity of mappings derived from differentiable functions and their relevance to the inverse function theorem.
5. To examine the role of Cauchy sequences in proving convergence and their connection to fixed points in complete metric spaces.

STRUCTURE:

12.1 INTRODUCTION:

The contraction principle, Differentiation of integrals
We now interrupt our discussion of differentiation to insert a fixed point theorem that is valid in arbitrary complete metric spaces. It will be used in the proof of the inverse function theorem.

12.2 DEFINITION:

Let X be a metric space with metric d.A mapping φ: X →X is said to be a Contraction of X into X and if there is a number $c < 1$ such that

$d\big(\varphi(x),\varphi(y)\big) \le cd(x,y)$for all $x, y \in X$

12.3 THEOREM:

(Contraction Mapping theorem or Fixed point theorem)

If X is a complete metric space, and if ϕ is a contraction of X into X, then there exists one and only $x \in X$ such that $\phi(x) = x$.

Proof: suppose X is a complete metric space and ϕ is as contraction of X into X.

claim:$\exists$ a unique $x \in X \Rightarrow \phi(x) = x$.

Suppose $x, y \in$ X such that $\phi(x) = x$and $\phi(y) = y$.

If possible suppose the $x \neq y$.

Since ϕ: X →X is a contraction, ∃ os real no. $c < 1$ Such that

$$d\big(\phi(p),\phi(q)\big) \le c \cdot d(p,q) \forall p, q \in X \qquad \to (1)$$

In particular, $0 < d(x,y) = d\big(\phi(x),\phi(y)\big) \le c \cdot d(x,y) < d(x,y)$

$$(\because c < 1).$$

$$\Rightarrow d(x,y) < d(x,y)$$

$$\therefore x = y$$

So, ϕ has a unique point.

First we show that ϕ had a fixed point.

Let $x_0 \in X$.

Define $\{x_n\}$ recursively, by setting $x_{n+1} = \phi(x_n)$ for $n = 0,1,2 \ldots \ldots$

So, from (1), $d(x_{n+1}, x_n) = d\big(\phi(x_n), \phi(x_{n-1})\big) \le cd(x_n, x_{n-1})$

for $n \geqslant 1$.

For $n = 1, d(x_2, x_1) \leqslant \mathrm{cd}(x_1, x_2)$.

For $n = 2, d(x_3, x_2) \le cd(x_2, x_1) \le c^2 d(x_1, x_0)$

By induction, that it follows that

$$d(x_{n+1}, x_n) \leqslant c^n d(x_1, x_0) \text{ for } n = 0,1,2, \ldots$$

For

$m > n \geqslant 1, d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots d(x_{m-1} x_n)$.

$\leqslant c^n d(x_1, x_0) + c^{n+1} d(x_1, x_0) + \cdots + c^{m-1} d(x_1, x_0)$.

$= (c^n + c^{n+1} + \cdots + c^{m-1}) d(x_1, x_0)$.

Now we show that $\{x_n\}$ is a cauchy sequence.

Let $\epsilon > 0$.

Since $1 + c + c^2 + \cdots$ is a convergent series, ∃ +ve integer N such that

$\sum_{k=n}^{m-1} C^k < \frac{\epsilon}{1+d(x_1,x_0)}$&$m \geqslant n \geqslant N$.

So, for$m \geqslant n \geqslant N$, $d(x_n, x_m) \leq (c^n + c^{n+1} + \cdots + c^{m-1})d(x_1, x_0)$

$$< \frac{\epsilon}{1 + d(x_1, x_0)} \cdot d(x_1, x_0) < \epsilon$$

$\therefore \{x_n\}$is a Cauchy sequence in X.

Since X is complete, the sequence $\{x_n\}$ converges to a point $x$in X, ie, $\lim_{n\to\infty} x_n = x$.

Claim: ϕ is Continuous on X.

Let $y \in X$ and $\epsilon > 0$. Put $\delta = \frac{\epsilon}{c}$

suppose $d(y, z) < \delta$ where$z \in x$.

Consider $d(\phi(y), \phi(z)) \leq cd(y, z) < c \cdot \delta = c \cdot \frac{\epsilon}{c} = \epsilon$.

$\therefore d(\phi(y), \phi(z)) < \epsilon$ When ever $d(y, z) < \delta\ \forall z \in X$.

This shows that ϕ is continuous at y.

$\therefore \phi$ is continuous on X.

So, since $x_n \to x, \phi(x_n) \to \phi(x)$

Hence, $\phi(x) = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} x_{n+1} = x$.

Thus ϕ has a fixed paint $x$in X.

Hence ϕ has a unique fixed point in x.

12.4 THE INVERSE FUNCTION THEOREM:

The inverse function theorem states, roughly speaking, that a continuously differentiable mapping f is invertible in a neighborhood of any point x at which the linear transformation $f'(x)$ is invertible.

Theorems on Inverse function Theorem

12.4.1 Theorem

Suppose **f** is a $\mathcal{C}'$-mapping of an open set $E \subset R^n$ into R^n, f'(a) is invertible for some $\bar{a} \in E, \bar{b} = \bar{f}(\bar{a})$. Then

(a) there exist open sets U and V in R^n such that $\bar{a} \in U, \bar{b} \in V \bar{f}$ is one-to-one on U, and

$$\bar{f}(U) = v$$

(b) If $\bar{g}$ is the inverse of $\bar{f}$ [which exists, by (a)], defined in V by

$\bar{g}\left(\bar{f}(\bar{x})\right) = \bar{x}(\bar{x} \in u)$ then $\bar{g} \in \mathcal{C}'(V)$.

proof:

Given$\bar{f}$ is $\mathcal{C}'$-mapping from E into R^n where E in to R^n where E is an open set in R^n,

$f^{-1}(\bar{a})$ is invertible and$\bar{b} = \bar{f}(\bar{a})$.

($\bar{f}$ is a $\mathcal{C}'$-mapping of E into R^n i.e $\bar{f}$is continuous differentiation

f^{-1} is a continuously mapping of E into $L(R^n, R^m)$)

Put $A = f^{-1}(\bar{a})$ Then A is invertible.

Put $\lambda = \frac{1}{2||A^{-1}||}$ Then $\lambda > 0$.

Since f^{-1} is continuous on E, f^{-1} is continuous at $\bar{a}$.

So$\exists$ a $\delta > 0$ such that $N_\delta(\bar{a}) \subseteq E$ and for any $\bar{x} \in N_\delta(\bar{a})$,

$\|f^{-1}(\bar{x}) - f^{-1}(\bar{a})\| < \lambda$.

Write $U = N_\delta(\bar{a})$.

Then U is a convex open subset of $E, \bar{a} \in U$ and $\|f^{-1}(\bar{x}) - A\| < \lambda$

for all $\bar{x} \in U$ $\rightarrow$ (1)

For each $\bar{y} \in R^n$,define as mapping $\phi: E \rightarrow R^n$ as

$\phi(\bar{x}) = \bar{x} + A^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right)$ for all $\bar{x} \in E$

Fix $\bar{y} \in R^n$.

For any $\bar{x} \in E$, Consider $\phi(\bar{x} + \bar{h}) - \phi(\bar{x}) - \left(I - A^{-1}f^{-1}(\bar{x})\right)\bar{h}$

$$= \bar{x} + \bar{h} + A^{-1}\left(\bar{y} - \bar{f}(\bar{x} + \bar{h})\right) - \left(\bar{x} + \bar{A}^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right) - \right.$$

$$-\left(I - \dot{A}^{-1} f^{-1}(\bar{x})\right)\bar{h}$$

$$= \bar{x} + \bar{h} + A^{-1}(\bar{y}) - A^{-1}\left(\bar{f}(\bar{x} + \bar{h})\right) - \bar{x} - A^{-1}(\bar{y}) + A^{-1}\left(\bar{f}(\bar{x})\right) - \bar{h}$$
$$+A^{-1}f^{-1}(\bar{x})\bar{h}$$

$$= \bar{x} + \bar{h} + A^{-1}(\bar{y}) - A^{-1}\left(f(\bar{x} + \bar{h})\right) - \quad \bar{x} - A^{-1}(\bar{y}) + A^{-1}(f(\bar{x}))\bar{h}.$$
$$+A^{-1}f^{-1}(\bar{x})\bar{h}$$

$$= -\bar{A}^{-1}(\bar{f}(\bar{x} + \bar{h}) - \bar{f}(\bar{x}) - f^{-1}(\bar{x})\bar{h})$$

$$\text{Consider } \lim_{h \to 0} \frac{\left|\phi(\bar{x} + \bar{h}) - \phi(\bar{x}) - \left(I - A^{-k}f^{-1}(\bar{x})\right)\bar{h}\right|}{|\bar{h}|}$$

$$= \lim_{h \to 0} \frac{| -A^{-1} \cdot \left(\bar{f}(\bar{x} + \bar{h}) - \bar{f}(\bar{x}) - f^{-1}(\bar{x})\bar{h} \right|}{|\bar{h}|}$$

$$\leq \sum\||A^{-1}\| \lim_{h \to 0} \frac{|\bar{f}(\bar{x} + \bar{h}) - \bar{f}(\bar{x}) - f^{-1}(\bar{x})\bar{h}|}{|\bar{h}|}$$

$$= \|A^{-1}\| \cdot 0 = 0$$

$$\therefore \lim_{h \to 0} \frac{\left|\phi(\bar{x}+\bar{h}) - \phi(\bar{x}) - \left(I - A^{-1}f^{-1}(\bar{x})\right)\bar{h}\right|}{\left|\bar{h}\right|} = 0$$

Hence, ϕ is differentiable at $\bar{x}$ and $\phi'(\bar{x}) = \mathrm{I} - \mathrm{A}^{-1}\mathrm{f}^{-1}(\bar{x}) \forall\, \bar{x} \in E$

$$\phi(\bar{x}) = \bar{x} \Leftrightarrow \bar{x} + A^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right) = \bar{x}$$

$$\Leftrightarrow A^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right) = 0$$

$$\Leftrightarrow \bar{y} - \bar{f}(\bar{x}) = 0$$

$$\Leftrightarrow \bar{f}(\bar{x}) = \bar{y}.$$

$$\therefore \phi(\bar{x}) \equiv \bar{x} \Leftrightarrow \bar{f}(\bar{x}) = \bar{y}$$

a) To prove part (a)

Now we show that $\bar{f}$ is one - to - one on U.

suppose $\bar{x}_1, \bar{x}_2 \in U$ such that $\bar{f}(\bar{x}_1) = \bar{f}(\bar{x}_2)$.

Let $\bar{y} = \bar{f}(\bar{x}_1) = \bar{f}(\bar{x}_2) \to$ (2)

for this $\bar{y}$, the function $\phi: E \to R^n$ defined by

$\phi(\bar{x}) = \bar{x} + A^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right) \forall \bar{x} \in E$ is differentiable,

$\phi'(\bar{x}) = I - A^{-1}\bar{f}^{-1}(\bar{x})$ and $\phi(\bar{x}) = \bar{x}$iff $\bar{y} = \bar{f}(\bar{x})$ for all $\bar{x} \in E. \to$ (3)

Consider $\phi'(\bar{x}) = I - \bar{A}^{-1}f^{-1}(\bar{x}) = A^{-1}A - \bar{A}^{-1}\mathrm{f}^{-1}(\bar{x})$

$$= A^{-1}\left(A - \mathrm{f}^{-1}(\bar{x})\right) = A^{-1}\left(f^{-1}(\bar{a}) - f^{-1}(\bar{x})\right).$$
$$\Rightarrow \|\phi'(\bar{x})\| = \|A^{-1}(f^{-1}(\bar{a}) - f^{-1}(\bar{x})\|$$
$$\leq \|A^{-1}\|\|f'(\bar{a}) - f'(\bar{x})\|$$
$$< \|A^{-1}\|\lambda = \frac{1}{2}\ \forall \bar{x} \in U.$$

$$\|\phi'(\bar{x})\| < \frac{1}{2}\ \forall \bar{x} \in U$$

we have $|\phi(\bar{x}) - \phi(\bar{z})| \leqslant \frac{1}{2}|\bar{x} - \bar{z}| \forall \bar{x}, \bar{z} \in U \to$ (4)

In particular $|\phi(\bar{x}_1) - \phi(\bar{x}_2)| \leq \frac{1}{2}|\bar{x}_1 - \bar{x}_2| \to$ (5)

But from (2)&(3), $\bar{y} = \bar{f}(\bar{x}_1), \bar{y} = \bar{f}(\bar{x}_2) \Rightarrow \phi(\bar{x}_1) = \bar{x}_1\ and \phi(\bar{x}_2) = \bar{x}_2$.

So, (5) $\Rightarrow |\bar{x}_1 - \bar{x}_2| \leq \frac{1}{2}|\bar{x}_1 - \bar{x}_2|$.

This is possible only when $\overline{x_1} = \overline{x_2}$.

$\therefore \bar{f}$is one- to-one on U.

put $V = \bar{f}(u)$.

Since $\bar{a} \in U, \bar{b} = \bar{f}(\bar{a}) \in V$.

Now we show that V is an open set.

Let $\bar{y}_0 \in V$. Then $\bar{y}_0 \in \bar{f}(u) \Rightarrow \bar{y}_0 = \bar{f}(\bar{x}_0)$ for some$\bar{x}_0 \in U$.

Let B be an open ball with centre $\bar{x}_0$ and radius $r > 0$ such that

$\bar{B} \subseteq$ U, where $\bar{B}$ is the closure of B.

Consider the open ball $N_{\lambda r}(\bar{y}_0)$.

To prove $N_{\lambda r}(\bar{y}_0) \subseteq V$.

Let $\bar{y} \in N_{\lambda r}(\bar{y}_0). \Rightarrow |\bar{y} - \bar{y}_0| < \lambda r$.

For this $\bar{y}$, we have a differentiable function $\phi: E \to \mathbb{R}^n$ defined by

$$\phi(\bar{x}) = \bar{x} + \mathrm{A}^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right) \forall \bar{x} \in E \text{ and } \phi'(\bar{x}) = \mathrm{I} - \mathrm{A}^{-1} f^{-1}(\bar{x}), \text{ and}$$

$$\phi(x) = \bar{x} \text{ iff } \bar{y} = \bar{f}(\bar{x})$$

$$\text{Now } |\phi(\bar{x}_0) - \bar{x}_0| = \left|\bar{x}_0 + \mathrm{A}^{-1}\left(\bar{y} - \bar{f}(\bar{x}_0)\right) - \bar{x}_0\right|$$

$$= \left|\bar{A}\left(\bar{y} - \bar{f}(\overline{x_0})\right)\right|$$

$$\leq \|\bar{A}\|\left|\bar{y} - \bar{f}(\bar{x}_0)\right|$$

$$= \|\bar{A}\||\bar{y} - \bar{y}_0|$$

$$< \|\mathrm{A}^{-1}\|\lambda r$$

$$= \frac{1}{2\lambda} \cdot \lambda r$$

$$= \frac{r}{2}$$

$$\therefore |\phi(\bar{x}_0) - \bar{x}_0| < \frac{r}{2} \to \quad (6)$$

Let $\bar{x} \in \bar{B}$. Then $\bar{x} \in U$.

Consider $|\phi(\bar{x}) - \bar{x}_0| = |\phi(\bar{x}) - \phi(\bar{x}_0) + \phi(\bar{x}_0) - \bar{x}_0|$

$$\leq |\phi(\bar{x}) - \phi(\bar{x}_0)| + |\phi(\bar{x}_0) - \bar{x}_0|$$

$$< \frac{1}{2}|\bar{x} - \bar{x}_0| + \frac{r}{2}(\text{ from (4)})$$

$$< \frac{r}{2} + \frac{r}{2} = r.$$

$$\Rightarrow |\phi(\bar{x}) - \bar{x}_0| < r$$

$$\Rightarrow \phi(\bar{x}) \in B$$

$$\therefore \bar{x} \in \bar{B} \Rightarrow \phi(\bar{x}) \in B$$

Since $\bar{B} \subseteq$ U, by (4), we have $|\phi(\bar{x}) - \phi(\bar{z})| \leqslant \frac{1}{2}|\bar{x} - \bar{z}| \forall \bar{x}, \bar{z} \in B$

for all $\bar{x}, \bar{z} \in \bar{B}$.

$\therefore \phi$ is a contraction of $\bar{\bar{B}}$ into $\bar{B}$.

Since $\bar{B}$ is a closed subset of R^n and R^n is complete, we have $\bar{B}$ is complete.

By Contraction mapping Theorem, ϕ has a fixed point, say $\bar{x} \in \bar{B}$.

$$\Rightarrow \phi(\bar{x}) = \bar{x} \Rightarrow \bar{y} = \bar{f}(\bar{x}).$$
$$\text{so, } \bar{y} = \bar{f}(\bar{x}) \in \bar{f}(\bar{B}) \subseteq \bar{f}\big((U)\big) = V$$
$$\Rightarrow \bar{y} \in V.$$
$$\therefore N_{\lambda r}(\bar{y}_0) \subseteq V$$

Hence V is an open subset of R^n.$|\phi(\bar{x}) - \phi(\bar{z})| \leqslant \frac{1}{2}|\bar{x} - \bar{z}| \forall \bar{x}, \bar{z} \in U$

Thus there exist open sets u add V in R^n such that $\bar{a} \in U, \bar{b} \in V$, $\bar{f}$ is one-to-one on u and $\bar{f}(u) = v$.

b) Suppose $\bar{g}$ is the inverse of $\bar{f}$ defined in V by

$$\bar{g}\left(\bar{f}(\bar{x})\right) = \bar{x} \quad \forall \bar{x} \in V$$

claim: $\bar{g}$ is a C'-mapping in V, i.e $\bar{g} \in C'(V)$.

First we show the g^{-1} exists in V.

Let $\bar{y} \in V$.

Since $V$is open, there exists $r > 0$ such that $S_r(\bar{y}) \subseteq V$.

Let $\bar{k} \in R^n$ such that $|\bar{k}| < r$.

Then $|\bar{y} + \bar{k} - \bar{y}| = |\bar{k}| < r$

$\Rightarrow \bar{y} + \bar{k} \in S_r(\bar{y}) \subseteq V \Rightarrow \bar{y} + \bar{k} \in V$

Since $\bar{y}, \bar{y} + \bar{k} \in V = f(U), \exists \bar{x}, \bar{z} \in U$ such that

$\bar{y} = \bar{f}(\bar{x})$and $\bar{y} + \bar{k} = \bar{f}(\bar{z})$.

Put $\bar{h} = \bar{z} - \bar{x}$.

Then $\bar{x} + \bar{h} = \bar{z} \in U$and $\bar{f}(\bar{x} + \bar{h}) = \bar{f}(\bar{z}) = \bar{y} + \bar{k}$.

So, $\bar{y} = \bar{f}(\bar{x})$ and $\bar{y} + \bar{k} = \bar{f}(\bar{x} + \bar{h}) \qquad \rightarrow \quad (7)$

For this $\bar{y}$, we get a differentiable function $\phi: E \rightarrow R^n$ defined by

$\phi(\bar{\omega}) = \bar{\omega} + A^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right)$ iff $\bar{y} = \bar{f}(\bar{\omega})$.

Consider $\phi(\bar{x} + \bar{h}) - \phi(\bar{x}) = \bar{x} + \bar{h} + A^{-1}\left(\bar{y} - \bar{f}(\bar{x} + \bar{h})\right) - \bar{x} - A^{-1}\left(\bar{y} - \bar{f}(\bar{x})\right)$

$$= \bar{x} + \bar{h} + A^{-1}(\bar{y}) - A^{-1}\bar{f}(\bar{x} + \bar{h}) - \bar{x} - A^{-1}(\bar{y}) + A^{-1}\bar{f}(\bar{x})$$
$$= \bar{h} + A^{-1}\left(\bar{f}(\bar{x}) - \bar{f}(\bar{x} + \bar{h})\right)$$
$$= \bar{h} + A^{-1}(\bar{y} - (\bar{y} + \bar{K}))$$

$$= \bar{h} - A^{-1}\bar{k}$$

$$\Rightarrow \phi(\bar{x} + \bar{h}) - \phi(\bar{x}) = \bar{h} - A^{-1}\bar{k}.$$

By (4), $|\phi(\bar{x} + \bar{h}) - \phi(\bar{x})| \leqslant \frac{1}{2}|\bar{x} + \bar{h} - \bar{x}| = \frac{1}{2}|\bar{h}|$

$$\Rightarrow |\phi(\bar{x} + \bar{h}) - \phi(\bar{x})| \leqslant \frac{1}{2}|\bar{h}|$$

$$\Rightarrow |\bar{h} - A^{-1}\bar{k}| \leq \frac{1}{2}|\bar{h}|$$

$$\Rightarrow \left||\bar{h}| - |A^{-1}\bar{k}|\right| \leq |\bar{h} - A^{-1}\bar{k}| \leq \frac{1}{2}|\bar{h}|$$

$$\Rightarrow \frac{-1}{2}|\bar{h}| \leq |\bar{h}| - |A^{-1}\bar{k}| \leq \frac{1}{2}|\bar{h}|$$

$$\Rightarrow -|\bar{h}| - \frac{1}{2}|\bar{h}| \leq -|A^{-1}\bar{k}| \leq \frac{1}{2}|\bar{h}| - |\bar{h}|$$

$$\Rightarrow -\frac{3}{2}|\bar{h}| \leq -|A^{-1}\bar{k}| \leq -\frac{1}{2}|\bar{h}|$$

$$\Rightarrow \frac{1}{2}|\bar{h}| \leq |A^{-1}\bar{k}| \leq \|A^{-1}\|\,|\bar{K}|$$

$$\Rightarrow |\bar{h}| \leq 2\ |\left|A^{-1}\|\,|\bar{k}\right| = \frac{1}{\lambda}\left|\bar{k}\right|$$

$$\Rightarrow |\bar{h}| \leq \frac{1}{\lambda}|\bar{k}| \qquad \rightarrow (8)$$

we have $\|f^{-1}(\bar{\omega}) - A\| < \lambda$ for $all\ \bar{\omega} \in U$.

So, for each $\|f^{-1}(\bar{\omega}) - A\|\|A^{-1}\| < \lambda \cdot \frac{1}{2\lambda} = \frac{1}{2} < 1$

$$\Rightarrow \|f^{-1}(\bar{\omega}) - A\|A^{-1}\| < 1 \text{ for all } \bar{\omega} \in U$$

So, by a known Theorem (4.8), $f^{-1}(\bar{\omega})$ is invertible for all $\bar{\omega} \in U$

In particular, $f^{-1}(\bar{x})$ is invertible:

Let the inverse of $f^{-1}(\bar{x})$ be T

Consider $\bar{g}(\bar{y} + \bar{k}) - \bar{g}(\bar{y}) - T\bar{k} = \bar{g}\left(\bar{f}(\bar{x} + \bar{h})\right) - \bar{g}\left(\bar{f}(\bar{x})\right) - T\bar{k}$

$$= -T(\bar{k} - f^{-1}(x)\bar{h})$$

$$= -T(\bar{f}(\bar{x} + \bar{h}) - \bar{f}(\bar{x}) - f^{-1}(\bar{x})\bar{h})(by)(7)$$

$$\Rightarrow \frac{|\bar{g}(\bar{y} + \bar{k}) - \bar{g}(\bar{y}) - T\bar{k}|}{|\bar{k}|} \leq \frac{\|T\|\,|\bar{f}(\bar{x} + \bar{h}) - \bar{f}(\bar{x}) - f^{-1}(\bar{x})\bar{h}|}{|\bar{k}|}$$

$$\leq \frac{\|T\|}{\lambda} \cdot \frac{|\bar{f}(\bar{x} + \bar{h}) - \bar{f}(\bar{x}) - f^{-1}(\bar{x})\bar{h}|}{|\bar{h}|} \qquad \rightarrow (9)$$

By (8) ,$\bar{h} \to \bar{0}$ os $\bar{k} \to \bar{0}$

Taking limit on both sides of (9) as $\bar{k} \to \bar{0}$, we get

$$\lim_{\bar{k}\to\bar{0}} \frac{\left|\bar{g}\left(\bar{y}+\bar{k}\right)-\bar{g}(\bar{y})-T\bar{k}\right|}{\left|\bar{k}\right|} \in 0$$

$$\text{i.e.} \frac{|\bar{g}(\bar{y}+\bar{k})-\bar{g}(\bar{y})-T\bar{k}|}{|\bar{k}|} = 0$$

So, $\bar{g}$is differentiable and $\bar{g}^{-1}(\bar{y}) = T$.

$$\Rightarrow \bar{g}^{-1}(\bar{y}) = \mathrm{T} = [f^{-1}(\bar{x})]^{-1} = \left[f^{-1}\left(\bar{g}(\bar{y})\right)\right]^{-1} \to \quad (10) \begin{pmatrix} \because\ \bar{f}(\bar{x}) = \bar{y} \text{ and} \bar{g} \\ \text{inverse of } \bar{f} \end{pmatrix}$$

Since every differentiable mapping is continuous, $\bar{g}$ is a Continous mapping of Vonto U.

Now $\bar{f}'$ is a Continuous mapping of U into Ω, where $\Omega$is the set of all invertible elements of $L(R^n)$; and the mapping $B \to B^{-1}$ of Ω into Ω is continuous on Ω.

∴ The inverse of $f^{-1}\left(\bar{g}(\bar{y})\right)$ is continuous.

So, by (10), $\bar{g}^{-1}$ is continuous on V, i.e$\bar{g} \in C'(V)$.

12.4.2 Theorem

If $\bar{f}$ is a C'-mapping of an open set $E \subset R^n$ into R^n and if $f^{-1}(\bar{x})$ is invertible for every$\bar{x} \in E$, then $\bar{f}(w)$ is an open subset of R^n for every open set $W \subset E$. [In other words, $\bar{f}$ is an open mapping of Einto R^n]

Proof: Suppose that$\bar{f}$ is a C'-mapping

Let W be any open subset of R^n such that $W \subset E$.

Claim;$\bar{f}(\omega)$ is an open subset of R^n.

Let $\bar{x} \in W \subset E$

$\Rightarrow \bar{x} \in E$.

By our supposition, $f^{-1}$is continuous at $\bar{x}$ and $f^{-1}(\bar{x})$ is invertible

So by part (a) of the Inverse function Theorem there exists open sets $U_{\bar{x}}$ in R^n such that $f: u_{\bar{x}} \to f(U_{\bar{x}})$ is one-to-one and $f(U_{\bar{x}})$ is open in R^n.

Now $W = \bigcup_{\bar{x}\in W} U_{\bar{x}}$ and that $\bar{f}(W) = \bigcup_{\bar{x}\in W} U_{\bar{x}}$

we know that union of open sets is open.

Hence, $\bar{f}(\omega)$ is an open subset of R^n for every open subset $W \subset E$.

Note: The hypotheses in this theorem ensure that each point $\bar{x} \in E$ has a neighborhood in which $\bar{f}$is one-to-one .This may be exposed by saying the $\bar{f}$ is locally one-to-one in E But $\bar{f}$need not be one-to-one in E under these circumstances.

12.5 THE IMPLICIT FUNCTION THEOREM:

If f is a continously differentiable real function in the plane, then the equation $f(x,y)=0$ can be solved for y in terms of x in a neighborhood of any point (a,b) at which $f(a,b)=0$ and $\frac{\partial f}{\partial y}\neq 0$. Likewise one can solve for x in terms of y near (a,b) if $\frac{\partial f}{\partial x}\neq 0$ at (a,b) For a simple example which illustrates the need for assuming $\frac{\partial f}{\partial y}\neq 0$.

Consider$f(x,y)=x^2+y^2-1$

12.5.1 Notations:

1) If $\bar{x}=(x_1,x_2,\rightarrow x_n)\in R^n$and $\bar{y}=(y_1,y_2,\dots,y_m)\in R^m$, then the point (or vector) $(x_1,x_2,\cdots,x_n,y_1,y_2,\dots,y_m)\in R^{n+m}$, and is denoted by $(\bar{x},\bar{y})$.

Thus $(\bar{x},\bar{y})=(x_1,x_2,\dots x_n,y_1,y_2,\dots,y_m)\in R^{n+m}$

In $(\bar{x},\bar{y})$ the first entry $\bar{x}$is a vector in R^n and the second entry$\bar{y}$ is a vector in R^m.

2) Every $A\in L(\mathrm{R}^{n+m},\mathrm{R}^n)$ can be split into two linear transformations $A_{\bar{x}}$ and $A_{\bar{y}}$, defined by $A_{\bar{x}}\bar{h}=A(\bar{h},\bar{0})$ and $A_{\bar{y}}\bar{K}=A(\bar{0},\bar{K})$ for ant any $\bar{h}\in R^n$ ad for any $\bar{k}\in \mathrm{R}^m$.

Then $A_{\bar{x}}\in L(R^n), A_{\bar{y}}\in L(R^m,R^n)$ and

$$A(\bar{h},\bar{k})=A_{\bar{x}}\bar{h}+A_{\bar{y}}\bar{k}\forall\bar{h}\in R^{\eta} \text{ and } \forall\bar{k}\in \mathrm{R}^m.$$

12.6 SUMMARY:

This lesson covers the contraction mapping theorem, which states that if a mapping is a contraction in a complete metric space, it has a unique fixed point. The proof involves showing that a generated sequence is Cauchy and converges. The lesson then discusses the inverse function theorem, indicating that a continuously differentiable mapping is locally invertible when the Jacobian is non-zero. It establishes that such mappings are one-to-one on a neighborhood and that their inverses are also continuous. Finally, the lesson briefly introduces the implicit function theorem for solving equations involving continuously differentiable functions.

12.7 TECHNICAL TERMS:

- Contraction mapping theorem
- Fixed point theorem
- Inverse function theorem
- Implicit function theorem

12.8 SELF ASSESSMENT QUESTIONS:

1. Show that the continuity of f' at the point a is needed in the inverse function theorem, even in the case n=1;If

$$f(t) = t + 2t^2 \sin(1/t)$$

For $t \neq 0$ and $f(0) = 0$ then $f'(0) = 1$, f' is bounded in (-1, 1) but f is not one-to-one in any neighborhood of 0.

2. Define a contraction mapping and state the contraction mapping theorem.
3. Prove the existence and uniqueness of a fixed point for a contraction mapping.
4. Explain the conditions under which the inverse function theorem applies.
5. Describe the significance of the Jacobian matrix in the inverse function theorem.
6. State and explain the implicit function theorem.

12.9 SUGGESTED READINGS:

1. Principles of Mathematics Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M. Apostal, Narosa Publishing House, 2nd Edition, 1985.

- **Dr. K. Bhanu Lakshmi**

LESSON- 13

THE IMPLICIT FUNCTION THEOREM

OBJECTIVES:

After reading this Lesson, the students should be able to:

- state and prove linear version of Implicit Function Theorem.
- state and prove Implicit Function Theorem for functions of several variables.

STRUCTURE:

13.1 INTRODUCTION:

In this lesson, two important theorems are introduced. First the linear version of the theorem is introduced and then the main theorem is discussed. Some of the basic definitions and notations are discussed before establishing the theorems.

If f is a continuously differentiable real function in the plane, then the equation $f(x, y) = 0$ can be solved for y in terms of x in a neighbourhood of any point (a, b) at which $f(a, b) = 0$ and $\frac{\partial f}{\partial y} \neq 0$. Likewise, one can solve for x in terms of y near (a, b) if $\frac{\partial f}{\partial y} \neq 0$ at (a, b).

For a simple example which illustrates the need for assuming $\frac{\partial f}{\partial y} \neq 0$,

consider $f(x, y) = x^2 + y^2 - 1$.

13.2 NOTATION:

1) If $\boldsymbol{x} = (x_1, x_2, x_{3,} \dots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, y_2, y_{3,} \dots, y_n) \in \mathbb{R}^n$, then the point (or vector) $(x_1, x_2, x_{3,} \dots, x_n, y_1, y_2, y_{3,} \dots, y_m) \in \mathbb{R}^{n+m}$, and is denoted by $(\boldsymbol{x}, \boldsymbol{y})$.

Thus $(\boldsymbol{x},\boldsymbol{y}) = (x_1, x_2, x_{3,}\dots, x_n, y_1, y_2, y_{3,}\dots, y_m) \in \mathbb{R}^{n+m}$.In $(\boldsymbol{x},\boldsymbol{y})$**,** the first entry $\boldsymbol{x}$ is a vector in $\mathbb{R}^n$ and the second entry $\boldsymbol{y}$ is a vector in $\mathbb{R}^m$.

2) Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into two linear transformations $A_{\boldsymbol{x}}$ and $A_{\boldsymbol{y}}$, defined by $A_{\boldsymbol{x}}\boldsymbol{h} = A(\boldsymbol{h}, \mathbf{0})$ *and* $A_{\boldsymbol{y}}\boldsymbol{k} = A(\mathbf{0}, \mathbf{k})$ for any $\boldsymbol{h} \in \mathbb{R}^n$and for any $\boldsymbol{k} \in \mathbb{R}^m$. Then $A_{\boldsymbol{x}} \in L(\mathbb{R}^n)$,$A_{\boldsymbol{y}} \in L(\mathbb{R}^m, \mathbb{R}^n)$ and
$A(\boldsymbol{h}, \mathbf{k}) = A_{\boldsymbol{x}}\boldsymbol{h} + A_{\boldsymbol{y}}\boldsymbol{k} \ \ \forall\, \boldsymbol{h} \in \mathbb{R}^n \ and \ \forall\, \mathbf{k} \in \mathbb{R}^m$.

13.3 THE LINEAR VERSION OF THE IMPLICIT FUNCTION THEOREM:

Theorem:

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if $A_{\boldsymbol{x}}$ is invertible, then there corresponds to every $\mathbf{k} \in \mathbb{R}^m$ a unique $\boldsymbol{h} \in \mathbb{R}^n$ such that $A(\boldsymbol{h}, \boldsymbol{k}) = \mathbf{0}$**.** This $\boldsymbol{h}$ can be computed from $\boldsymbol{k}$ by the formula $\boldsymbol{h} = -(A_x)^{-1}A_y\mathbf{k}$

Proof: Let $\mathbf{k} \in \mathbb{R}^m$ then $A_y\mathbf{k} \in \mathbb{R}^m$

Since A_x is invertible, ${A_x}^{-1}$ exists.

Put $\mathbf{h} = -(A_x)^{-1}A_y\mathbf{k}$

Clearly $\mathbf{h} \in \mathbb{R}^n$

Consider $A(\boldsymbol{h}, \boldsymbol{k}) \ = A_x\mathbf{h} + A_y\mathbf{k}$

$$= A_x\big(-(A_x)^{-1}A_y\mathbf{k}\big) + A_y\mathbf{k}$$

$$= -A_x(A_x)^{-1}A_y\mathbf{k} + A_y\mathbf{k}$$

$$= -A_y\mathbf{k} + A_y\mathbf{k} = \mathbf{0}$$

Thus for $\mathbf{k} \in \mathbb{R}^m$ there exists $\mathbf{h} \in \mathbb{R}^n$ such that $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$

Uniqueness: Suppose $\mathbf{h_1}, \mathbf{h_2} \in \mathbb{R}^n$ such that $A(\mathbf{h_1}, \mathbf{k}) = \mathbf{0}$ and $A(\mathbf{h_2}, \mathbf{k}) = \mathbf{0}$

$\Rightarrow A_x\mathbf{h_1} + A_y\mathbf{k} = 0 \ and \ A_x\mathbf{h_2} + A_y\mathbf{k} = 0$

$\Rightarrow {A_x}^{-1}\big(A_x\mathbf{h_1} + A_y\mathbf{k}\big) = 0 \ and \ {A_x}^{-1}\big(A_x\mathbf{h_2} + A_y\mathbf{k}\big) = 0$

$\Rightarrow \mathbf{h_1} + {A_x}^{-1}\big(A_y\mathbf{k}\big) = 0 \ and \ \mathbf{h_2} + {A_x}^{-1}\big(A_y\mathbf{k}\big) = 0$

$\Rightarrow \mathbf{h_1} = -(A_x)^{-1}A_y\mathbf{k} = \mathbf{h_2}$

Hence, there exists a unique $\mathbf{h} \in \mathbb{R}^n$ such that $A(\mathbf{h}, \mathbf{k}) = 0$

13.4. THE IMPLICIT FUNCTION THEOREM:

13.4.1 Statement: Let **f** be a $\mathscr{C}'$-mapping of an open set $E \subseteq \mathbb{R}^{n+m}$ into $\mathbb{R}^n$, such that $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{0}$ for some point $(\mathbf{a},\mathbf{b}) \in E$.

Put $A = \mathbf{f}'(\mathbf{a},\mathbf{b})$ and assume that $A_{\mathbf{x}}$ is invertible. Then there exist open sets $U \subset \mathbb{R}^{n+m}$ and $W \subset \mathbb{R}^m$, with $(\mathbf{a},\mathbf{b}) \in U$ and $\mathbf{b} \in W$, having the following property:

To every $\mathbf{y} \in W$ corresponds a unique $\mathbf{x}$ such that

$(\mathbf{x},\mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{0}$ (I)

If this $\mathbf{x}$ is defined to be $\mathbf{g}(\mathbf{y})$, then

$\mathbf{g}$ is a $\mathscr{C}'$-mapping of W into $\mathbb{R}^n$, $\mathbf{g}(\mathbf{b}) = \mathbf{a}$,

$\mathbf{f}(\mathbf{g}(\mathbf{y}),\ \mathbf{y}) = \mathbf{0}$ $(\mathbf{y} \in W)$, $\mathbf{g}'(\mathbf{b}) = -(A_{\mathbf{x}})^{-1}A_{\mathbf{y}}$ (II)

Proof: Define $\mathbf{F}: E \to \mathbb{R}^{n+m}$ as $\mathbf{F}(\mathbf{x},\ \mathbf{y}) = (\mathbf{f}(\mathbf{x},\mathbf{y}),\ \mathbf{y})\ \forall\ (\mathbf{x},\ \mathbf{y}) \in E$

As $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{0},\ \mathbf{F}(\mathbf{a},\mathbf{b}) = (\mathbf{f}(\mathbf{a},\mathbf{b}),\ \mathbf{b}) = (\mathbf{0},\mathbf{b})$ (1)

Since **f** is a $\mathscr{C}'$-mapping on E, it follows that **F** is also a $\mathscr{C}'$-mapping.

To prove that $\mathbf{F}'(\mathbf{a},\mathbf{b})$ is an invertible element of $L(\mathbb{R}^{n+m})$

Since **f** is differentiable at $(\mathbf{a},\mathbf{b})$, we have

$\mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}) - \mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{f}'(\mathbf{a},\mathbf{b})(\mathbf{h},\mathbf{k}) + \boldsymbol{\gamma}(\mathbf{h},\mathbf{k})$ where

$\lim_{(\mathbf{h},\mathbf{k})\to\mathbf{0}} \left|\frac{\boldsymbol{\gamma}(\mathbf{h},\mathbf{k})}{(\mathbf{h},\mathbf{k})}\right| = \mathbf{0}$ where $\boldsymbol{\gamma}$ is the remainder that occurs in the definition of $\mathbf{f}'(\mathbf{a},\mathbf{b})$

$\Rightarrow \mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}) = \mathbf{A}(\mathbf{h},\mathbf{k}) + \boldsymbol{\gamma}(\mathbf{h},\mathbf{k})$ (2) $(\because\ A = \mathbf{f}'(\mathbf{a},\mathbf{b})\ and\ \mathbf{f}(\mathbf{a},\mathbf{b}) = 0)$.

Consider $\boldsymbol{F}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}) - \boldsymbol{F}(\mathbf{a},\mathbf{b})$

$$= (\mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}),\mathbf{b}+\mathbf{k}) - (\mathbf{f}(\mathbf{a},\mathbf{b}),\mathbf{b})$$

$$= (\mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}),\mathbf{b}+\mathbf{k}) - (0,\mathbf{b})$$

$$= (\mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}),\mathbf{b}) + (\mathbf{0},\mathbf{k}) - (\mathbf{0},\mathbf{b})$$

$$= (\mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}),\mathbf{0}) + (\mathbf{0},\mathbf{k})$$

$$= (\mathbf{f}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}),\mathbf{k})$$

$= (\mathbf{A}(\mathbf{h},\mathbf{k}) + \boldsymbol{\gamma}(\mathbf{h},\mathbf{k}), \mathbf{k})$ (from (2))

$= (\mathbf{A}(\mathbf{h},\mathbf{k}), \mathbf{k}) + (\gamma(\mathbf{h},\mathbf{k}), \mathbf{0})$

$\Rightarrow \boldsymbol{F}(\mathbf{a}+\mathbf{h}, \mathbf{b}+\mathbf{k}) - \boldsymbol{F}(\mathbf{a},\mathbf{b}) - (\mathbf{A}(\mathbf{h},\mathbf{k}), \mathbf{k}) = (\boldsymbol{\gamma}(\mathbf{h},\mathbf{k}), \mathbf{0})$

$\Rightarrow \lim_{(\mathbf{h},\mathbf{k})\to \mathbf{0}} \frac{|\boldsymbol{F}(\mathbf{a}+\mathbf{h},\mathbf{b}+\mathbf{k}) - \boldsymbol{F}(\mathbf{a},\mathbf{b}) - (\mathbf{A}(\mathbf{h},\mathbf{k}), \boldsymbol{k})|}{|(\mathbf{h},\mathbf{k})|}$

$= \lim_{(\mathbf{h},\mathbf{k})\to 0} \frac{|(\boldsymbol{\gamma}(\mathbf{h},\mathbf{k}), \mathbf{0})|}{|(\mathbf{h},\mathbf{k})|} = 0$

So, $\mathbf{F}'(\mathbf{a},\mathbf{b})$ is a linear operator on $\mathbb{R}^{n+m}$ that maps $(\mathbf{h},\mathbf{k})$ to $(\mathbf{A}(\mathbf{h},\mathbf{k}), \mathbf{k})$.

To prove $\mathbf{F}'(\mathbf{a},\mathbf{b})$ is one-to-one

$\mathbf{F}'(\mathbf{a},\mathbf{b})(\mathbf{h},\mathbf{k}) = \mathbf{0}$ where $(\mathbf{h},\mathbf{k}) \in \mathbb{R}^{n+m}$

$\Leftrightarrow (\mathbf{A}(\mathbf{h},\mathbf{k}), \mathbf{k}) = (\mathbf{0}, \mathbf{0})$

$\Leftrightarrow \mathbf{A}(\mathbf{h},\mathbf{k}) = \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$

$\Leftrightarrow A_{\mathbf{x}}\mathbf{h} + A_{\mathbf{y}}\mathbf{k} = \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$ [$\because$ A $\in$L ($\mathbb{R}^{n+m}, \mathbb{R}^{n}$) can be split into linear transformations $A_{\mathbf{x}}$ and A_{y} defined by $A_{\mathbf{x}}\mathbf{h} = \mathbf{A}(\mathbf{h}, \mathbf{0})$, $A_{\mathbf{y}}\mathbf{k} = \mathbf{A}(\mathbf{0}, \mathbf{k})$ $\forall$ $\mathbf{h} \in \mathbb{R}^{n}$, $\mathbf{k} \in \mathbb{R}^{m}$]

$\Leftrightarrow A_{\mathbf{x}}\mathbf{h} = \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$

$\Leftrightarrow \mathbf{h} = \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$ ($\because$ $A_{\mathbf{x}}$ is invertible)

Therefore $\mathbf{F}'(\mathbf{a},\mathbf{b})$ is one-to-one

By a known theorem (9.5) , $\mathbf{F}'(\mathbf{a},\mathbf{b})$ is onto

Hence, $\mathbf{F}'(\mathbf{a},\mathbf{b})$ is invertible on $\mathbb{R}^{n+m}$

Therefore **F** satisfies all conditions of Inverse mapping theorem.

So, by the Inverse mapping theorem, there exist open sets U and V in $\mathbb{R}^{n+m}$ such that

$(\mathbf{a},\mathbf{b}) \in U, \boldsymbol{F}(\boldsymbol{a},\boldsymbol{b}) \in V$ (3)

$\boldsymbol{F}(U) = V$ and **F** is a one-to-one mapping of U onto V.

Write $W = \{\mathbf{y} \in \mathbb{R}^{m} / (\mathbf{0}, \mathbf{y}) \in V\}$

From (1) & (3), $(\mathbf{0}, \mathbf{b}) \in V \Rightarrow \mathbf{b} \in$ W

Now we show that W is an open set in $\mathbb{R}^{m}$

Let $\mathbf{y} \in \mathrm{W}$. Then $(\mathbf{0}, \mathbf{y}) \in \mathrm{V}$

Since V is an open set in $\mathbb{R}^{n+m}$, there exists $\delta > 0$ such that $N_\delta(\mathbf{0}, \mathbf{y}) \subseteq \mathrm{V}$

Consider $N_\delta(\mathbf{y})$, which is a neighbourhood in $\mathbb{R}^m$.

Let $\mathbf{h} \in N_\delta(\mathbf{y})$

$\Rightarrow |\mathbf{h} - \mathbf{y}| < \delta$

Now $(\mathbf{0}, \mathbf{h}) \in \mathbb{R}^{n+m}$ and $|(\mathbf{0}, \mathbf{h}) - (\mathbf{0}, \boldsymbol{y})| = |\mathbf{h} - \mathbf{y}| < \delta$

$$\Rightarrow (\mathbf{0}, \mathbf{h}) \in N_\delta(\mathbf{0}, \boldsymbol{y}) \subseteq \mathrm{V}$$

$$\Rightarrow (\mathbf{0}, \mathbf{h}) \in \mathrm{V} \Rightarrow \mathbf{h} \in \mathrm{W}$$

Since $\mathbf{h} \in N_\delta(\mathbf{y})$ is arbitrary, $N_\delta(\mathbf{y}) \subseteq \mathrm{W}$

Therefore W is an open set in $\mathbb{R}^m$.

To Prove (I): Let $\mathbf{y} \in \mathrm{W}$. Then $(\mathbf{0}, \boldsymbol{y}) \in \mathrm{V}$

$$\Rightarrow (\mathbf{0}, \boldsymbol{y}) \in \mathbf{F}(\mathrm{U})$$

$$\Rightarrow (\mathbf{0}, \boldsymbol{y}) = \mathbf{F}(\mathbf{x}, \mathbf{z}) \text{ for some } (\mathbf{x}, \mathbf{z}) \in \mathrm{U}$$

$$\Rightarrow (\mathbf{0}, \boldsymbol{y}) = (\mathbf{f}(\mathbf{x}, \mathbf{z}), \mathbf{z})$$

$$\Rightarrow \mathbf{f}(\mathbf{x}, \mathbf{z}) = \mathbf{0} \text{ and } \mathbf{z} = \mathbf{y}$$

$$\Rightarrow \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$$

So, for $\mathbf{y} \in \mathrm{W}$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that $(\mathbf{x}, \mathbf{y}) \in \mathrm{U}$ and $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$

Uniqueness: Suppose $\mathbf{f}(\mathbf{x}_1, \mathbf{y}) = \mathbf{0}$ and $\mathbf{f}(\mathbf{x}_2, \mathbf{y}) = \mathbf{0}$ where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$

Now $\mathbf{F}(\mathbf{x}_1, \mathbf{y}) = (\mathbf{f}(\mathbf{x}_1, \mathbf{y}), \mathbf{y}) = (\mathbf{0}, \boldsymbol{y}) = (\mathbf{f}(\mathbf{x}_2, \mathbf{y}), \mathbf{y}) = \boldsymbol{F}(\mathbf{x}_2, \mathbf{y})$

Therefore $\mathbf{F}(\mathbf{x}_1, \mathbf{y}) = \mathbf{F}(\mathbf{x}_2, \mathbf{y})$

$\Rightarrow \mathbf{x}_1 = \mathbf{x}_2$ $\qquad$ $(\because \mathbf{F} \text{ is one} - \text{one})$

Thus there exists unique $\mathbf{x} \in \mathbb{R}^n$ such that $(\mathbf{x}, \mathbf{y}) \in \mathrm{U}$ and $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.

To Prove (II):

Define $\mathbf{g}: \mathrm{W} \to \mathbb{R}^n$ as follows:

Let $\mathbf{y} \in W$

Then by the proof given above there exists unique $\mathbf{x} \in \mathbb{R}^n$

such that $(\mathbf{x}, \mathbf{y}) \in \mathbf{U}$ and $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$

Now define $\mathbf{g}(\mathbf{y}) = \mathbf{x}$

For $\mathbf{y} \in W$, $(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{x}, \mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ and so

$\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y})$

Therefore $\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y})$ for all $\mathbf{y} \in W$

Define $\mathbf{G}: V \to U$ as $\mathbf{G}(\mathbf{F}(\mathbf{z})) = \mathbf{z}$ for all $\mathbf{z} \in U$

Then $\mathbf{G}$ is the inverse of $\mathbf{F}$

So, by Inverse function theorem, $\mathbf{G} \in \mathscr{C}'(V)$

Since $\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y})$ for all $\mathbf{y} \in W$, we have

$(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{F}^{-1}(\mathbf{0}, \mathbf{y}) = \boldsymbol{G}(\mathbf{0}, \mathbf{y})$ for all $\mathbf{y} \in W$.

As $\mathbf{G} \in \mathscr{C}'(V)$, $\mathbf{g} \in \mathscr{C}'(W)$.

Also, $(\mathbf{g}(\mathbf{b}), \mathbf{b}) = \mathbf{G}(\mathbf{0}, \mathbf{b}) = \mathbf{F}^{-1}(\mathbf{0}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})$ (by (1))

$\Rightarrow (\mathbf{g}(\mathbf{b}), \mathbf{b}) = (\mathbf{a}, \mathbf{b})$

$\Rightarrow \mathbf{g}(\mathbf{b}) = \mathbf{a}$

Now we show that $\mathbf{g}'(\mathbf{b}) = -(\mathbf{A_x})^{-1}\mathbf{A_y}$

Define $\emptyset: W \to \mathbb{R}^{n+m}$ as $\emptyset(\mathbf{y}) = (\mathbf{g}(\mathbf{y}), \mathbf{y}) \quad \forall \quad \boldsymbol{y} \in W$

Then $\emptyset(\mathbf{b}) = (\mathbf{g}(\mathbf{b}), \mathbf{b}) = (\mathbf{a}, \mathbf{b})$ and $\mathbf{f}(\emptyset(\mathbf{y})) = \mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0} \ \forall \ \boldsymbol{y} \in W$

Now $\emptyset'(\mathbf{y})\mathbf{k} = (\mathbf{g}'(\mathbf{y})\mathbf{k}, \mathbf{k}) \quad \forall \quad \boldsymbol{y} \in W$ and $\mathbf{k} \in \mathbb{R}^m$ (4)

By chain Rule, $\mathbf{f}'(\emptyset(\mathbf{y}))\emptyset'(\mathbf{y}) = \mathbf{0} \quad \forall \quad \boldsymbol{y} \in W$ (5)

Now $\emptyset(\mathbf{b}) = (\mathbf{g}(\mathbf{b}), \mathbf{b}) = (\mathbf{a}, \mathbf{b})$

When $\mathbf{y} = \mathbf{b}$, equation (5) becomes

$\mathbf{f}'(\emptyset(\mathbf{b}))\emptyset'(\mathbf{b}) = \mathbf{0}$

$\Rightarrow \mathbf{f}'(\mathbf{a},\mathbf{b})\emptyset'(\mathbf{b}) = \mathbf{0}$

$\Rightarrow A\emptyset'(\mathbf{b}) = \mathbf{0}$ $(\because \mathbf{b} \in W)$ (6)

$\Rightarrow A\emptyset'(\mathbf{b})(\mathbf{k}) = \mathbf{0} \quad \forall \quad \mathbf{k} \in \mathbb{R}^m$

$\Rightarrow A(\mathbf{g}'(\mathbf{b})\mathbf{k}, \mathbf{k}) = \mathbf{0} \quad \forall \quad \mathbf{k} \in \mathbb{R}^m$

$\Rightarrow A_{\mathbf{x}}\mathbf{g}'(\mathbf{b})\mathbf{k} + A_{\mathbf{y}}\mathbf{k} = \mathbf{0} \quad \forall \quad \mathbf{k} \in \mathbb{R}^m$

$\Rightarrow A_{\mathbf{x}}\mathbf{g}'(\mathbf{b}) + A_{\mathbf{y}} = \mathbf{0}$

$\Rightarrow A_{\mathbf{x}}\mathbf{g}'(\mathbf{b}) = -A_{\mathbf{y}}$

$\Rightarrow \mathbf{g}'(\mathbf{b}) = -(A_{\mathbf{x}})^{-1}A_{\mathbf{y}}$

Hence the theorem.

13.4.2 Example: The following is an example for the Implicit Function Theorem and find $\mathbf{g}'$

Take $n = 2, m = 3$

Consider the mapping $\mathbf{f} = (f_1, f_2)$ of $\mathbb{R}^5$ into $\mathbb{R}^2$,

given by $f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2y_1 - 4y_2 + 3$ and

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2\cos x_1 - 6x_1 + 2y_1 - y_3$$

Put $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (3, 2, 7)$

Then $f_1(\mathbf{a}, \mathbf{b}) = f_1(0, 1, 3, 2, 7) = 2e^0 + 1x3 - 4x2 + 3 = 0$

$f_2(\mathbf{a}, \mathbf{b}) = f_2(0,1,3,2,7) = 1.cos0 - 6x0 + 2x3 - 7 = 0$

So, $\mathbf{f}(\mathbf{a}, \mathbf{b}) = (f_1(\mathbf{a}, \mathbf{b}), f_2(\mathbf{a}, \mathbf{b})) = (0,0)$

Put $A = \mathbf{f}'(\mathbf{a}, \mathbf{b})$

With respect to the standard bases, the matrix of the transformation A is given by

$$[A] = \begin{bmatrix} D_1f_1 & D_2f_1 & D_3f_1 & D_4f_1 & D_5f_1 \\ D_1f_2 & D_2f_2 & D_3f_2 & D_4f_2 & D_5f_2 \end{bmatrix}_{at(\mathbf{a},\ \mathbf{b})}$$

$$[A] = \begin{bmatrix} 2e^{x_1} & y_1 & x_2 & -4 & 0 \\ -6 - x_2\sin x_1 & \cos x_1 & 2 & 0 & -1 \end{bmatrix}_{at(0,\ 1,\ 3,\ 2,\ 7)}$$

$= \begin{bmatrix} 2 & 31 & -4 & 0 \\ -6 & 12 & 0 & -1 \end{bmatrix}$

Hence $[\mathrm{A_x}] = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$ and $[\mathrm{A_y}] = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$

It is clear that $\mathrm{A_x}$ is invertible and

$[(\mathrm{A_x})^{-1}] = [\mathrm{A_x}]^{-1} = \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$

Now $[\mathbf{g}'(\mathbf{b})] = [\mathbf{g}'(3,2,7)] = -[\mathrm{A}_x]^{-1}[\mathrm{A_y}]$

$= -\frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/5 & -3/20 \\ -1/2 & 6/5 & 1/10 \end{bmatrix}$

In terms of partial derivatives,

$\mathrm{D}_1 g_1 = 1/4 \quad \mathrm{D}_2 g_1 = 1/5 \quad \mathrm{D}_3 g_1 = -3/20$

$\mathrm{D}_1 g_2 = -1/2 \quad \mathrm{D}_2 g_2 = 6/5 \quad \mathrm{D}_3 g_2 = 1/10$ at the point (3, 2, 7).

13.5 SUMMARY:

In this lesson we have discussed two theorems: Linear version of the implicit function theorem and the implicit function theorem.
The implicit function theorem gives the condition, under which an implicit relationship between variables can be expressed in an explicit manner.

13.6 TECHNICAL TERMS:

- Explicit function
- Implicit function theorem

13.7 SELF – ASSESSMENT QUESTIONS:

1. Can $f(x, y) = x^3 + y^3 - 2xy$ be expressed by an explicit function $y = g(x)$ in a neighbourhood of the point (1,1)?

2. Check whether theorem 13.4.1 can be applied at all points, where $x^2 - y^2 = 0$.

13.8 SUGGESTED READINGS:

1. Principles of Mathematical Analysis by Walter Rudin, 3rd Edition.
2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985.

- Dr. K. Siva Prasad.

LESSON- 14
DETERMINANTS

OBJECTIVES:

After reading this lesson, the students should be able to

- understand the concepts of determinant of a matrix of a linear operator A on $\mathbb{R}^n$ and Jacobian.
- Prove $\det[I] = 1$ where I is the identity operator on $\mathbb{R}^n$.
- Prove $\det[A]_1 = -\det[A]$ where $[A]_1$ is obtained from [A] by interchanging two columns.
- Prove $\det([B][A]) = \det[B]\det[A]$ for any two nxn matrices [A] and [B].
- Prove a linear operator A on $\mathbb{R}^n$ is invertible if and only if $\det[A] \neq 0$.

STRUCTURE:

14.1 INTRODUCTION:

In this lesson we define the determinant of the matrix of a linear operator on $\mathbb{R}^n$, and also we discuss properties of the determinant.

Determinants are numbers associated to square matrices, and hence they are numbers associated to linear operators represented by such matrices.

It is 0 if and only if the corresponding operator fails to be invertible.

14.2. DEFINITION:

If $(j_1, j_2, \dots\dots, j_n)$ is an ordered n-tuple of integers,

define $s(j_1, j_2, \dots\dots, j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p)$ where $\operatorname{sgn}x = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

Thus $s(j_1, j_2, \ldots\ldots, j_n) = 1, -1$ or 0 and it changes sign if any two of the j's are interchanged.

14.2.1 Example:

s(2, 3, 1) = sgn(1-3).sgn(1-2).sgn(3-2)

= sgn(-2).sgn(-1).sgn(1)

= (-1).(-1).1 =1

14.2.2 Definition:

Let [A] be the matrix of a linear operator A on $\mathbb{R}^n$ relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots\ldots, \mathbf{e}_n\}$ with entries a(i, j) or a_{ij} in the i^{th} row and j^{th} column. The determinant denoted by det[A] is defined as the number

$$\det[A] = \Sigma s(j_1, j_2, \ldots, j_n)\ a(1, j_1)a(2, j_2) \ldots . a(n, j_n) \qquad (1)$$

The sum in (1) extends over all ordered n-tuples of integers $(j_1, j_2, \ldots\ldots, j_n)$with $1 \le j_r \le n$

The column vectors $\mathbf{x}_j$ of [A] are $\mathbf{x}_j = \sum_{i=1}^{n} a(i, j)\mathbf{e}_i \quad (1 \le j \le n)$

It is convenient to think of det[A] as a function of the column vectors of [A]. If we write $\det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \det[A]$, det is a real valued function defined over the set of all ordered n-tuples of vectors in $\mathbb{R}^n$.

14.2.3 Example:

If $[A] = \begin{bmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \end{bmatrix}$, then det[A] = s(1, 1) a(1, 1) a(2, 1) + s(1, 2) a(1, 1) a(2, 2)

+ s(2, 1) a(1, 2) a(2, 1) + s(2, 2) a(1, 2) a(2, 2)

= a(1, 1) a(2, 2) - a(1, 2) a(2, 1)

14.3. THEOREMS:

14.3.1 THEOREM:

a) If I is the identity operator on $\mathbb{R}^n$, then det[I]= $\det(\mathbf{e}_1, \mathbf{e}_2, \ldots\ldots, \mathbf{e}_n) = 1$
b) det is a linear function of each of the column vectors $\mathbf{x}_j$, if the others are held fixed.
c) If $[A]_1$ is obtained from [A] by interchanging two columns, then $\det[A]_1 = -\det[A]$.
d) If [A] has two equal columns, then det[A] = 0.

Proof:

a) Let A be the identity operator on $\mathbb{R}^n$, i.e, A = I.

Consider the matrix [I].

In [I], the i^{th} row j^{th} column entry, $a(i,j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

So, det[I]= $\Sigma s(j_1, j_2, \ldots\ldots, j_n)\, a(1,j_1)a(2,j_2)\ldots . a(n,j_n)$

$= s(1, 2, \ldots\ldots, n)a(1,1)a(2,2)\ldots . a(n,n)$

=1.

Therefore $\det[I] = 1$

Also, by the definition, $\det[I] = \det(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n)$ where each $\mathbf{x}_i$ is a column vector of I.

That is $\det[I] = \det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n) = 1$.

b) By the definition, $s(j_1, j_2, \ldots\ldots, j_n) = 0$ if any two of the j's are equal. Each of the remaining n! products in the summation det[A] contains exactly one factor from each column.

Therefore $\det[A]$ is a linear function of each of the column vectors $\mathbf{x}_j$.

c) Let $[A]_1$, be a matrix, obtained from[A] by interchanging two columns. Then $s(j_1, j_2, \ldots\ldots, j_n)$ changes sign.

Therefore $\det[A]_1 = -\det[A]$.

d) Suppose [A] has two equal columns.

If we interchange the two equal columns, there is no change in [A]

So, by (c), $\det[A] = -\det[A] \Rightarrow 2\det[A] = 0 \Rightarrow \det[A] = 0$.

14.3.2 THEOREM:

If [A] and [B] are n by n matrices, then $\det([B][A]) = \det[B]\det[A]$

Proof: Suppose [A] and [B] are nxn matrices.

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be the column vectors of [A].

Define $\Delta_B(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) = \Delta_B[A] = \det([B][A])$ (1)

Since the columns of [B][A] are the vectors $B\mathbf{x}_1, B\mathbf{x}_2, \ldots B\mathbf{x}_n$, we have $\Delta_B(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n) = \det(B\mathbf{x}_1, B\mathbf{x}_2, \ldots B\mathbf{x}_n)$ (2)

By (2) and Theorem (14.3.1), Δ_B also has the following properties:

i) $\Delta_B = \det$ is a linear function of each of the column vectors $B\mathbf{x}_j$.

ii) If $([B][A])_1$ is obtained from $[B][A]$ by interchanging two columns, then
$\det([B][A])_1 = -\det([B][A])$ (3)

iii) If $[B][A]$ has two equal columns, then $\det([B][A]) = 0$

Since $\mathbf{x}_j = \sum_{i=1}^{n} a(i, j)\mathbf{e}_i$, and $\Delta_B = \det$ is a linear function of each of the column vectors,

we have $\Delta_B[A] = \Delta_B(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$

$\Rightarrow \Delta_B[A] = \Delta_B(\sum_{i=1}^{n} a(i, 1)\mathbf{e}_i, \mathbf{x}_2, \ldots \mathbf{x}_n)$

$\Rightarrow \Delta_B[A] = \sum_{i=1}^{n} a(i, 1)\Delta_B(\mathbf{e}_i, \mathbf{x}_2, \ldots \mathbf{x}_n)$

Also, $\mathbf{x}_2 = \sum_{i=1}^{n} a(i, 2)\mathbf{e}_i$

So $\Delta_B[A] = \sum_{i=1}^{n}(a_i, 1)\,\Delta_B\left(\mathbf{e}_i, \sum_{i=1}^{n}(a_{i_2}, 2)\mathbf{e}_i, \mathbf{x}_3, \ldots, \mathbf{x}_n\right)$

$= \sum_{i=1}^{n}(a_i, 1)\,(a_{i_2}, 2)\Delta_B\left(\mathbf{e}_i, \mathbf{e}_{i_2}, \mathbf{x}_3, \ldots, \mathbf{x}_n\right)$

Repeating this process with $\mathbf{x}_3, \mathbf{x}_4, \ldots, \mathbf{x}_n$, we have

$\Delta_B[A] = \Sigma a(i_1, 1).a(i_2, 2)\ldots a(i_n, n)\Delta_B\left(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}\right)$ (4)

when the sum is extended over all ordered n-tuples$(i_1, i_2, \ldots, i_n)$ with $i \leq i_r \leq n$ But we have

$\Delta_B\left(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \ldots, \mathbf{e}_{i_n}\right) = t(i_1, i_2, \ldots, i_n)\Delta_B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ where $t = 1, 0, -1$. (5)

Substituting (5) in (4), we get

$\Delta_B[A] = \{\Sigma a(i_1, 1).a(i_2, 2)\ldots a(i_n, n)t(i_1, i_2, \ldots, i_n)\}\Delta_B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ (6)

Since $[B][I] = [B]$, by (1), we have $\Delta_B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = \det[B]$ (7)

Using (7) in (6), we get $\det([B][A]) = \Delta_B[A]$

$\det([B][A]) = \{\Sigma a(i_1, 1).a(i_2, 2)\ldots a(i_n, n)t(i_1, i_2, \ldots, i_n)\}\det[B]$ (8)

for all nxn matrices [A] and [B]

Taking B = I in (8), we get

$\Delta_I[A] = \{\Sigma a(i_1, 1).a(i_2, 2)\ldots a(i_n, n)t(i_1, i_2, \ldots, i_n)\}\det[I]$

$\det[A] = \{\Sigma a(i_1, 1).a(i_2, 2)\ldots a(i_n, n)t(i_1, i_2, \ldots, i_n)\}(1)$

$(\because \Delta_I[A] = \det([I][A]) = \det[A])$

So, (8) becomes $\Delta_B[A] = \det[A]\det[B]$.

14.3.3 THEOREM:

A linear operator A on $\mathbb{R}^n$ is invertible if and only if $\det[A] \neq 0$

Proof: Let A be a linear operator on $\mathbb{R}^n$

Suppose that A is invertible.

Then by the above theorem,

$\det[A]\det[A^{-1}] = \det[AA^{-1}] = \det[I] = 1,$

so that $\det[A] \neq 0$.

Conversely, suppose that $\det[A] \neq 0$

Claim: A is invertible.

On the contrary suppose that A is not invertible

Then the column vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ of [A] are linearly dependent

So, there is one $\mathbf{x}_k$ with $1 \leq k \leq n$ such that

$$\mathbf{x}_k + \sum_{j \neq k} c_j \mathbf{x}_j = 0 \text{ for some scalar } c_j,\ 1 \leq j \leq n \ \& \ j \neq k \qquad (1)$$

By a known theorem (14.3.1) the det is a linear function of each of the column vectors $\mathbf{x}_j$,

if the others held fixed, and

det[A] = 0 if [A] has two equal columns.

So, $\mathbf{x}_k$ can be replaced by $\mathbf{x}_k + c_j\mathbf{x}_j$ without changing the

determinant, if $j \neq k$.

Repeating, we see that $\mathbf{x}_k$ can be replaced by the left side of (1)

i.e., by 0, without changing the determinant.

But a matrix which has **0** for one column has determinant 0.

Therefore $\det[A] = 0$

Hence, A is invertible.

14.4 REMARK:

Suppose $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ are bases in $\mathbb{R}^n$.

Every linear operator A on $\mathbb{R}^n$ determines matrices [A] and $[A]_u$, with entries a_{ij} and α_{ij}, given by

$A\mathbf{e}_j = \sum_i a_{ij}\mathbf{e}_i$ and $A\mathbf{u}_j = \sum_i \alpha_{ij}\mathbf{u}_i$

Let B be an invertible linear operator on $\mathbb{R}^n$. Suppose $[B] = [\,b_{ij}]$

If $\mathbf{u}_j = B\mathbf{e}_j = \sum_i b_{ij}\mathbf{e}_i$, then the matrix [B] is invertible and

$$A\mathbf{u}_j = \sum_k \alpha_{kj} \sum_i b_{ik}\mathbf{e}_i$$

$$= \sum_i \sum_k (b_{ik}\alpha_{kj})\mathbf{e}_i$$

Also, $A\mathbf{u}_j = AB\mathbf{e}_j = A\sum_k b_{kj}\mathbf{e}_k = \sum_i \left(\sum_k a_{ik}b_{kj}\right)\mathbf{e}_i$.

Therefore $\sum_k b_{ik}\alpha_{kj} = \sum_k a_{ik}b_{kj}$.

$\Rightarrow [B][A]_u = [A][B]$ (1)

Since B is invertible, $\det[B] \neq 0$.

From (1), we have $\det([B][A]_u) = \det([A][B])$

$$\Rightarrow \det[B]\,.\det[A]_u = \det[A].\det[B]$$

$$\Rightarrow \det[A] = \det[A]_u.$$

Therefore the determinant of the matrix of a linear operator does not depend on the basis which is used to construct the matrix.

Hence, it is meaningful to speak of the determinant of a linear operator, without having any basis in mind.

14.5 JACOBIANS:

If **f** maps an open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$, and if **f** is differentiable at a Point $\mathbf{x} \in E$, the determinant of the linear operator $\mathbf{f}'(\mathbf{x})$ is called the Jacobian of **f** at **x**

In symbols, $J_{\mathbf{f}}(\mathbf{x}) = \det \mathbf{f}'(\mathbf{x})$.

Notation: We write $\frac{\partial(y_1,\ y_2,\ \dots\dots,y_n)}{\partial(x_1,\ x_2,\ \dots\dots,x_n)}$ for $J_{\mathbf{f}}(\mathbf{x})$, if $(y_1,\ y_2,\ \dots\dots,y_n) = \mathbf{f}(x_1,\ x_2,\ \dots,x_n)$.

In terms of Jacobians, the hypothesis in the inverse function theorem is that $J_{\mathbf{f}}(\boldsymbol{a}) \neq 0$. If the implicit function theorem is stated in terms of the functions

$\mathbf{f_1}(x_1, x_2, \dots, x_n, y_1, y_2, \dots\dots, y_m) = 0$

$\mathbf{f_2}(x_1, x_2, \dots, x_n, y_1, y_2, \dots\dots, y_m) = 0$

$\mathbf{f_n}(x_1, x_2, \dots, x_n, y_1, y_2, \dots\dots, y_m) = 0$

The assumption made there on A amounts to $\frac{\partial(\mathbf{f_1}, \mathbf{f_2}, \dots\dots, \mathbf{f_n})}{\partial(x_1, x_2, \dots\dots, x_n)} \neq 0$

14.6 SUMMARY:

In this lesson we have defined the determinant of the matrix of a linear operator on $\mathbb{R}^n$, and we have discussed related theorems and examples as well as the term "Jacobian".

14.7 TECHNICAL TERMS:

- Matrix of a linear operator
- Determinant of a matrix
- Jacobian

14.8 SELF- ASSESSMENT QUESTIONS:

1. Find s(2, 3, 2)

2. Prove that the determinant of a linear operator A on $\mathbb{R}^n$ is independent from the choices of basis on $\mathbb{R}^n$.

14.9 SUGGESTED READINGS:

1. Principles of Mathematical Analysis by Walter Rudin, 3rd Edition.

2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985.

- Dr. K. Siva Prasad.

LESSON- 15

DERIVATIVES OF HIGHER ORDER AND DIFFERENTIATION OF INTEGRALS

OBJECTIVES:

After reading this lesson, the students should be able to

- understand the concept of second order partial derivatives of a real function defined in an open set $E \subseteq \mathbb{R}^n$.
- state and prove mean value theorem for real functions of two variables
- understand that under what conditions on $\emptyset$ can one prove that the equation

  $\frac{d}{dt}\int_a^b \phi(x,\ t)dx = \int_a^b \frac{\partial \phi}{\partial t}(x,\ t)dx$ is true, where $\emptyset$ is a function of two variables which can be differentiated with respect to the other.

STRUCTURE:

15.1 INTRODUCTION:

In this lesson we define second order partial derivatives of a real function of two variables and we established two theorems. We shall first discuss the mean- value theorem for real functions of two variables and also we shall discuss another theorem on a function $\emptyset$ of two variables which can be integrated with respect to one and which can be differentiated with respect to the other.

15.2. DEFINITIONS:

15.2.1 Definition: Suppose f is a real function defined in an open set $E \subseteq \mathbb{R}^n$, with partial derivatives $D_1 f,\ D_2 f,\ \dots\ , D_n f$. If the functions $D_j f$ are themselves differentiable, then the second-order partial derivatives of f are defined by

$D_{ij} f = D_i D_j f$ (i,j=1, 2, 3, …. ,n).

If all these functions $D_{ij}f$ are continuous in E, we say that f is of class $\mathscr{C}''$ in E, or that **f** $\in \mathscr{C}''$ (E).

15.2.2 Definition: A mapping **f** of E into $\mathbb{R}^m$ is said to be of class $\mathscr{C}''$ if each component of **f** **is** of class $\mathscr{C}''$.

Note:

1) $D_{ij}(\boldsymbol{f}) = D_i(D_j \boldsymbol{f}) = \frac{\partial^2 \boldsymbol{f}}{\partial x_i \partial x_j}$
2) $D_{ij}(\boldsymbol{f})$ and $D_{ji}(\boldsymbol{f})$ need not be the same.

15.3. THEOREM (MEAN- VALUE THEOREM):

15.3.1 Statement: Suppose f is defined in an open set $E \subseteq \mathbb{R}^2$, and $D_1 f$ and $D_{21} f$ exist at every point of E. Suppose $Q \subseteq E$ is a closed rectangle with sides parallel to the coordinate axes, having (a, b) and $(a + h,\ b + k)$ as opposite vertices ($h \neq 0,\ k \neq 0$).

put $\Delta(f,\ Q) = f(a + h,\ b + k) - f(a + h,\ b) - f(a,\ b + k) + f(a,\ b)$.

Then there is a point (x, y) in the interior of Q such that

$\Delta(f,\ Q) = hk(D_{21} f)(x,\ y)$.

Proof: Suppose f is a real function defined in E where E is an open set in $\mathbb{R}^2$

Put $\Delta(f,\ Q) = f(a + h,\ b + k) - f(a + h,\ b) - f(a,\ b + k) + f(a,\ b)$

For $t \in [a,\ a + h]$, put $u(t) = f(t,\ b + k) - f(t,\ b)$ (1)

Then 'u' is continuous on $[a, a + h]$ and differentiable in $(a, a + h)$

So, by a known theorem () there exists $x \in (a, a + h)$ such that

$u(a + h) - u(a) = (a + h - a)u'(x) = hu'(x)$ (2)

Note that $D_1 f = \frac{\partial f}{\partial x} = u'(x)$ (3)

And $D_{21}f = \frac{\partial}{\partial y}(D_1 f)$

Since $D_{21}f$ exists, $D_1 f$ is differentiable in $(b,\ b+k)$ and continuous on $[b,\ b+k]$

So, by a known theorem () there exists $y \in (b,\ b+k)$ such that

$D_1 f(x,\ b+k) - D_1 f(x,\ b) = (b+k-b) D_{21} f(x,\ y)$

$$= k D_{21} f(x,\ y) \qquad (4)$$

Consider $\Delta(f,\ Q) = f(a+h,\ b+k) - f(a+h,\ b) - f(a,\ b+k) + f(a+b)$

$$=u(a+b) - u(a) \qquad \text{(by (1))}$$

$$= hu'(x)$$

$$= h[(D_1 f)(x, b+k) - (D_1 f)(x, b)]$$

$$= hk D_{21} f(x,\ y)$$

$\Rightarrow \Delta(f,\ Q) = hk D_{21} f(x,\ y)$.

But $(x,\ y)$ is a point in the interior of Q.

Therefore there exists a point $(x,\ y)$ in the interior of Q such that

$\Delta(f, Q) = hk D_{21} f(x, y)$

15.3.2 Theorem: Suppose f is defined in an open set $E \subseteq \mathbb{R}^2$, suppose that $D_1 f,\ D_{21} f$, and $D_2 f$ exist at every point of E, and $D_{21} f$ is continuous at some point $(a, b) \in E$. Then $D_{12} f$ exists at $(a,\ b)$ and $(D_{12} f)(a,\ b) = (D_{21} f)(a,\ b)$.

Proof: Suppose f is a real function defined in E, where E is an open set in $\mathbb{R}^2$.

Also suppose that $D_1 f,\ D_{21} f$, and $D_2 f$ exist at every point of E, and $D_{21} f$ is continuous at some point $(a,\ b) \in E$.

Put $A = (D_{21} f)(a,\ b)$.

Let $\varepsilon > 0$ be given

Since $D_{21} f$ is continuous at $(a,\ b)$, there exists $\delta > 0$ such that

$|(D_{21} f)(a,\ b) - (D_{21} f)(x,\ y)| < \varepsilon$ whenever $|(a,\ b) - (x, y)| < \delta\ \ \forall\ (x,\ y) \in E$

$\Rightarrow |A - (D_{21} f)(x,\ y)| < \varepsilon$ whenever $|(a,\ b) - (x,\ y)| < \delta$ (1)

Choose h and k such that $|h| < \frac{\delta}{2}$ and $|k| < \frac{\delta}{2}$.

Let $Q \subseteq E$ be the closed rectangle with sides parallel to the coordinate axes having $(a,\ b)$ and $(a+h,\ b+k)$ as opposite vertices $(h \neq 0,\ k \neq 0)$.

Put $\Delta(f,\ Q) = f(a+h,\ b+k) - f(a+h,\ b) - f(a,\ b+k) + f(a,\ b)$

So, the above theorem, there exists a point (x, y) in the interior of Q such that

$\Delta(f, Q) = hk(D_{21})f(x,\ y)$

$\Rightarrow (D_{21}f)(x,\ y) = \frac{\Delta(f,\ Q)}{hk}$ (2)

From (1) & (2), $|A - (D_{21}f)(x,\ y)| < \varepsilon \Rightarrow \left|\frac{\Delta(f,Q)}{hk} - A\right| < \varepsilon$

Fix h.

As $k \to 0$ i.e., $b+k \to b,\ f(a+h,\ b+k) \to f(a+h,\ b)$ and

$f(a,\ b+k) \to f(a,\ b)$

Since D_2f exists in E, we have $\lim_{k\to 0}\left|\frac{\Delta(f,Q)}{hk} - A\right| < \varepsilon$

$\Rightarrow \left|\frac{(D_2f)(a+h,b)-(D_2f)(a,b)}{h} - A\right| < \varepsilon$

Since ε is arbitrary and the above inequality holds for all h with $|h| < \frac{\delta}{2}$, we have that

$(D_{12}f)(a,b) = A$

$\Rightarrow (D_{21}f)(a,b) = (D_{12}f)(a,b)$

15.3.3 COROLLARY: $D_{21}f = D_{12}f$ if $f \in \mathscr{C}''(E)$

Proof: Suppose $f \in \mathscr{C}''(E)$

Then f is a real function defined in the open set $E \subseteq \mathbb{R}^2$, with partial derivatives $D_1f,\ D_2f$ which are differentiable in E and the 2nd order partial derivatives $D_{ij}f, 1 \leq i,j \leq 2$ are continuous in E. (1)

Let $(a,b) \in E$.

From (1), $D_{21}f$ is continuous at (a,b)

Now we prove that

$$(D_{12}f)(a,b) = (D_{21}f)(a,b).$$

Suppose f is a real function defined in E, where E is an open set in $\mathbb{R}^2$

Put $A = (D_{21}f)(a,\ b)$.

Let $\varepsilon > 0$ be given

Since $D_{21}f$ is continuous at $(a,\ b)$, there exists $\delta > 0$ such that

$|(D_{21}f)(a,\ b) - (D_{21}f)(x,\ y)| < \varepsilon$ whenever $|(a,\ b) - (x,\ y)| < \delta \quad \forall (x,\ y) \in E$

$\Rightarrow |A - (D_{21}f)(x,\ y)| < \varepsilon$ whenever $|(a,b) - (x,\ y)| < \delta$ (1)

Choose h and k such that $|h| < \frac{\delta}{2}$ and $|k| < \frac{\delta}{2}$.

Let $Q \subseteq E$ be the closed rectangle with sides parallel to the coordinate axes having $(a,\ b)$ and $(a+h,\ b+k)$ as opposite vertices $(\mathrm{h} \neq 0, \mathrm{k} \neq 0)$.

Put $\Delta(f,\ Q) = f(a+h,\ b+k) - f(a+h,b) - f(a,\ b+k) + f(a,\ b)$

By the above theorem, there exists a point (x,y) in the interior of Q such that

$\Delta(f,Q) = hk(D_{21}f)(x,\ y)$

$\Rightarrow (D_{21}f)(x,\ y) = \frac{\Delta(f,\ Q)}{hk}$ (2)

From (1) & (2), $|A - (D_{21}f)(x,\ y)| < \varepsilon \Rightarrow \left|\frac{\Delta(f,Q)}{hk} - A\right| < \varepsilon$

Fix h.

As $k \to 0$ i.e., $b+k \to b$, $f(a+h,\ b+k) \to f(a+h,\ b)$ and

$f(a,\ b+k) \to f(a,\ b)$

Since $D_2 f$ exists in E, we have $\lim_{k\to 0}\left|\frac{\Delta(f,\ Q)}{hk} - A\right| < \varepsilon$

$\Rightarrow \left|\frac{(D_2f)(a+h,\ b) - (D_2f)(a,\ b)}{h} - A\right| < \varepsilon$

Since ε is arbitrary and the above inequality holds for all h with $|h| < \frac{\delta}{2}$, we have that

$(D_{12}f)(a,\ b) = A$

$\Rightarrow (D_{12}f)(a,\ b) = (D_{21}f)(a,\ b)$

This is true for every $(a,\ b) \in E$

Therefore $D_{12}f = D_{21}f$

15.4 DIFFERENTIATION OF INTEGRALS:

Suppose $\emptyset$ is a function of two variables which can be integrated with respect to one and which can be differentiated with respect to the other. Under what condition on $\emptyset$ can one prove that the equation

$\frac{d}{dt}\int_a^b \phi(x,\ t)dx = \int_a^b \frac{\partial\phi}{\partial t}(x,\ t)dx$ is true?

Notation: It is convenient to use the notation $\phi^t(x) = \phi(x,\ t)$.

Thus, for each t, ϕ^t is a function of one variable.

We recall the following theorem.

15.4.1 Theorem: Let α be monotonically increasing on $[a,\ b]$.

Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ for n=1,2,… and suppose $f_n \to f$ uniformly on $[a,\ b]$.

Then $f \in \mathcal{R}(\alpha)$ on $[a,\ b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty}\int_a^b f_n d\alpha$.

15.4.2 Theorem: Suppose

(a) $\phi(x,\ t)$ is defined for $a \le x \le b, c \le t \le d$;
(b) α is an increasing function on $[a, b]$;
(c) $\phi^t \in \mathcal{R}(\alpha)$ for every $t \in [c, d]$;
(d) $c < s < d$, and to every $\varepsilon > 0$ corresponds a $\delta > 0$

such that $|(D_2\phi)(x,\ t) - (D_2\phi)(x,\ s)| < \varepsilon$ for all $x \in [a,\ b]$ and for all

$t \in (s - \delta,\ s + \delta)$.

Define $f(t) = \int_a^b \phi(x,\ t)d\alpha(x) \quad (c \le t \le d)$

Then $(D_2\phi)^s \in \mathcal{R}(\alpha), f'(s)$ exists, and $f'(s) = \int_a^b (D_2\phi)(x,\ s)\, d\alpha(x)$.

Proof: Suppose (a), (b), (c) & (d)

For any $t \in [c,\ d]$,

define $f(t) = \int_a^b \phi(x,\ t) d\alpha(x)$

Now we prove that

$(D_2\phi)^s \in \mathcal{R}(\alpha), f'(s)$ exists, and $f'(s) = \int_a^b (D_2\phi)(x,\ s)\, d\alpha(x)$

Consider the difference quotients

$\psi(x,\ t) = \frac{\phi(x,\ t) - \phi(x,\ s)}{t-s}$ for $0 < |t - s| < \delta$ (1)

Since $D_2\phi$ exists, ϕ is differentiable in $(s,\ t)$ and continuous on $[s,\ t]$, by a known result (Lagrange's mean-value theorem applied to 2^{nd} variable of ϕ) there corresponds to each $(x,\ t)$ a number 'u' between s and t such that $\phi(x,\ t) - \phi(x,\ s) = (t - s)D_2\phi(x,\ u)$.

$\Rightarrow \frac{\phi(x,\ t) - \phi(x,\ s)}{t-s} = D_2\phi(x,\ u)$

$\Rightarrow \psi(x,\ t) = D_2\phi(x,\ u)$ (by (1))

By our supposition (d), we have

$|\psi(x,\ t) - D_2\phi(x,\ s)| = |D_2\phi(x,\ u) - D_2\phi(x,\ s)| < \varepsilon$ for all $a \le x \le b$ and

$0 < |t - s| < \delta$.

i.e., $\lim_{t \to s} \psi(x,\ t) = D_2\phi(x,\ s)$ uniformly on $[a,\ b]$ (2)

Consider

$\frac{f(t) - f(s)}{t-s} = \int_a^b \frac{\phi(x,\ t)}{t-s} d\alpha(x) - \int_a^b \frac{\phi(x,\ s)}{t-s} d\alpha(x)$

$= \int_a^b \frac{\phi(x,t) - \phi(x,s)}{t-s} d\alpha(x)$

$= \int_a^b \psi(x,t) d\alpha(x)$ (by (1)) (3)

$= \int_a^b \psi^t(x) d\alpha(x)$

By (2), $\psi^t \to (D_2\emptyset)^s$, uniformly on $[a, b]$ as $t \to s$.

By our supposition (c), $\emptyset^t \in \mathcal{R}(\alpha)$ for all $t \in [c,\ d]$

So, $\psi^t \in \mathcal{R}(\alpha)$ for all $t \in [c,d]$.

Therefore by Theorem () $(D_2\emptyset)^s \in \mathcal{R}(\alpha)$ and

$$\int_a^b (D_2\emptyset)^s(x)d\alpha(x) = \lim_{t\to s}\int_a^b \psi^t\, d\alpha(x)$$

$$= \lim_{t\to s}\frac{f(t)-f(s)}{t-s} \qquad \text{(by (3))}$$

$$= f'(s)$$

Hence, $(D_2\emptyset)^s \in \mathcal{R}(\alpha)$ and $f'(s)$ exists and $f'(s) = \int_a^b (D_2\phi)(x,\ s)\, d\alpha(x)$

15.5 SUMMARY:

In this lesson we have defined second order partial derivatives of a real function of two variables and we have discussed related theorems and also we have discussed one theorem related to differentiation of integrals.

15.6 TECHNICAL TERMS:

- Differentiation of integrals
- Mean-value theorem

15.7 SELF- ASSESSMENT QUESTIONS:

1. Prove analogues of theorem 15.4.2 with $(-\infty,\infty)$ in the place of $[a,b]$.

2. Put $f(0,0) = 0$ and $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ if $(x,y) \neq (0,0)$. Prove that

(i) $f, D_1f,\ D_2f$ are continuous in $\mathbb{R}^2$;

(ii) $D_{12}f$ and $D_{21}f$ exists at every point of $\mathbb{R}^2$, and are continuous except at (0,0);

(iii) $(D_{12}f)(0,0) = 1$ and $(D_{21}f)(0,0) = -1$.

15.8 SUGGESTED READINGS:

1. Principles of Mathematical Analysis by Walter Rudin, 3rd Edition.

2. Mathematical Analysis by Tom M.Apostal, Narosa Publishing House, 2nd Edition, 1985.

- Dr. K. Siva Prasad.