

(DM21)

ASSIGNMENT - 1

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2019

Second Year

MATHEMATICS

Topology and Functional Analysis

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) Define topological space and give an example.  
Let  $\tau_1$  and  $\tau_2$  be two topologies on a nonempty set  $X$ . Show that  $\tau_1 \cap \tau_2$  is also a topology on  $X$ .
- b) Let  $f : X \rightarrow Y$  be a mapping of one topological space into another. Show that  $f$  is continuous if and only if  $f^{-1}(F)$  is closed in  $X$  whenever  $F$  is closed in  $Y$  if and only if  $f(\overline{A}) \subseteq \overline{f(A)}$  for every subset  $A$  of  $X$ .
- Q2)** a) Prove that any closed subspace of a compact space is compact.  
b) State and prove Heine – Borel theorem.
- Q3)** a) Prove that a metric space is sequentially compact if and only if it has the Bolzano – Weierstrass property.  
b) Prove that a metric space is compact if and only if it is complete and totally bounded.
- Q4)** a) Prove that in a Hausdorff space, any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighborhoods.  
b) State and prove Tietze extension theorem.
- Q5)** a) Prove that a subspace of the real line  $\mathbf{R}$  is connected if and only if it is an interval.  
b) Let  $X$  be a topological space and  $A$  a connected subspace of  $X$ . Prove that if  $B$  is a subspace  $X$  such that  $A \subseteq B \subseteq \overline{A}$  then  $B$  is connected. In particular, show that  $\overline{A}$  is connected.

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ASSIGNMENT - 2

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2019

Second Year

MATHEMATICS

Topology and Functional Analysis

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) If  $N$  and  $N'$  are normed linear spaces then prove that the set  $\mathcal{B}(N, N')$  of all continuous linear transformations of  $N$  into  $N'$  is itself a normed linear space with respect to the pointwise linear operations and the norm defined by  $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$ .
- Further, show that if  $N'$  is a Banach space, then prove that  $\mathcal{B}(N, N')$  is also a Banach space.
- b) i) If  $M$  is a closed linear subspace of a normed linear space  $N$  and  $x_0$  is a vector not in  $M$ , then prove that there exists a functional  $f_0$  in  $N^*$  such that  $f_0(M) = 0$  and  $f_0(x_0) \neq 0$ .
- ii) If  $N$  is a normed linear space then prove that there exists a natural imbedding of  $N$  in  $N^{**}$ .
- Q2)** a) State and prove the closed graph theorem.
- b) State and prove the uniform boundedness theorem.
- Q3)** a) Prove that a closed convex subset  $C$  of a Hilbert space  $H$  contains a unique vector of smallest norm.
- b) Prove that if  $M$  is a closed linear subspace of a Hilbert space  $H$  then prove that  $H = M \oplus M^\perp$ .
- Q4)** a) State and prove Gram-Schmidt orthogonalization process.
- b) Define the adjoint of an operator  $T$  on a Hilbert space  $H$  and prove that the mapping  $T \mapsto T^*$  is a mapping of  $\mathcal{B}(H)$  into itself.
- Q5)** a) Define a projection and a perpendicular projection. Let  $M$  be a closed linear subspace of a Hilbert space  $H$ . Then prove that  $P$  is the projection on  $M$  if and only if  $I-P$  is the projection on  $M^\perp$ . Also, prove that if  $P$  is a projection on  $M$  then
- $$x \in M \Leftrightarrow Px = x \Leftrightarrow \|Px\| = \|x\|$$
- b) If  $P_1, P_2, \dots, P_n$  are the projections on closed linear subspaces  $M_1, M_2, \dots, M_n$  of a Hilbert space  $H$  then prove that  $P = P_1 + P_2 + \dots + P_n$  is a projection if and only if the  $P_i$ 's are pairwise orthogonal; and in this case,  $P$  is the projection on  $M = M_1 + M_2 + \dots + M_n$ .

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**ASSIGNMENT - 1**  
**M.Sc. (Second) DEGREE EXAMINATION, MAY – 2019**  
**Second Year**  
**MATHEMATICS**

**Measure and Integration**

**MAXIMUM : 30 MARKS**

**ANSWER ALL QUESTIONS**

- Q1)** a) Let  $A$  be a countable set. Prove that the set of all finite sequences from  $A$  is also countable.
- b) State axiom of choice and apply it to prove the following:  
“Let  $f: X \rightarrow Y$  be a mapping on to  $Y$ . Then there is a mapping  $g: Y \rightarrow X$  such that  $f \circ g$  is the identity map on  $Y$ ”.
- Q2)** a) Prove that the interval  $(a, \infty)$  is measurable.  
b) Prove that the collection  $\mathcal{M}$  of measurable sets is a  $\sigma$  – algebra.
- Q3)** a) Let  $C$  be a constant and  $f$  and  $g$  two measurable real – valued functions defined on the same domain. Then prove that the functions  $f + c$ ,  $cf$ ,  $f + g$ ,  $g - f$ , and  $fg$  are also measurable.  
b) Let  $E$  be a measurable set of finite measure, and  $\{f_n\}$  a sequence of measurable functions defined on  $E$ . Let  $f$  be a real – valued function such that for each  $x$  in  $E$  we have  $f_n(x) \rightarrow f(x)$ . Then prove that given  $\epsilon > 0$  and  $\delta > 0$  there is a measurable set  $A \subset E$  with  $m(A) < \delta$  and an integer  $N$  such that for all  $x \notin A$  and all  $n \geq N$ , we have  $|f_n(x) - f(x)| < \epsilon$ .
- Q4)** a) Let  $\varphi$  and  $\psi$  be simple functions which vanish outside a set of finite measure. Then prove that  $\int (a\varphi + b\psi) = a \int \varphi + b \int \psi$ , and if  $\varphi \geq \psi$  a.e. then  $\int \varphi \geq \int \psi$ .  
b) Let  $f$  be a nonnegative function which is integrable over a set  $E$ . Then prove that given  $\epsilon > 0$  there is a  $\delta > 0$  such that for every set  $A \subset E$  with  $m(A) < \delta$  we have  $\int_A f < \epsilon$ .
- Q5)** a) State and prove Lebesgue convergences theorem.  
b) Show that if  $f$  is integrable over  $E$ , then so is  $|f|$  and  $\left| \int_E f \right| \leq \int_E |f|$ . Does the integrability of  $|f|$  imply that of  $f$ ?

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ASSIGNMENT - 2

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2019

Second Year

MATHEMATICS

Measure and Integration

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) State and prove Vitali lemma.  
b) Prove that if  $f$  is bounded and measurable on  $[a, b]$  and  $F(x) = \int_a^x f(t)dt + F(a)$ , then  $F'(x) = f(x)$  for almost all  $x$  in  $[a, b]$ .
- Q2)** a) State and prove Holder inequality.  
b) Prove that a normed linear space  $X$  is complete if and only if every absolutely summable series summable.
- Q3)** a) Let  $(X, \mathcal{B}, \mu)$  be a measure space. Then prove that there exists a complete measure space  $(X, \mathcal{B}_0, \mu_0)$  such that  
i)  $\mathcal{B} \subset \mathcal{B}_0$   
ii)  $E \in \mathcal{B} \Rightarrow \mu(E) = \mu_0(E)$ , and  
iii)  $E \in \mathcal{B}_0 \Leftrightarrow E = A \cup B$  where  $B \in \mathcal{B}$  and  $A \subset C, C \in \mathcal{B}$  with  $\mu(C) = 0$ .  
b) Let  $\nu$  be a signed measure on the measurable space  $(X, \mathcal{B})$ . Then prove that there are two mutually singular measures  $\nu^+$  and  $\nu^-$  on  $(X, \mathcal{B})$  such that  $\nu = \nu^+ - \nu^-$ . Moreover, show that there is only one such pair of mutually singular measures.
- Q4)** State and prove the Radon – Nikodym theorem.
- Q5)** a) Define measure on an algebra  $\mathcal{A}$  and prove that if  $A \in \mathcal{A}$  then  $A$  is measurable with respect to  $\mu^*$ .  
b) State and prove Carathéodory extension theorem.



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ASSIGNMENT - 1

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2019

Second Year

MATHEMATICS

Analytical Number Theory and Graph Theory

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) State and prove Euler's summation formula.  
b) For all  $x \geq 1$  show that  $\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O(\sqrt{x})$  for  $C$  the Euler's constant.
- Q2)** a) For  $x > 1$  show that  $\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$ .  
b) For all  $x \geq 1$  show that  $\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1$  with equality holding only if  $x < 2$ .
- Q3)** a) State and prove Abel's identity.  
b) For  $x \geq 2$  show that  $\theta(x) = \Pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$ .
- Q4)** a) Let  $F$  be a real or complex valued function defined on  $(0, \infty)$ , and  $G(x) = \log x \cdot \sum_{n \leq x} F\left(\frac{x}{n}\right)$  then show that  $F(x) \log x + \sum_{n \leq x} F\left(\frac{x}{n}\right) \wedge (n) = \sum_{d \leq x} \mu(d) G\left(\frac{x}{d}\right)$ .  
b) State and prove Selberg's asymptotic formula.
- Q5)** a) If a graph has exactly two vertices of odd degree show that there is a path joining these vertices.  
b) Show that a connected graph  $G$  is an Euler graph if and only if it can be decomposed into circuits.

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ASSIGNMENT - 2

M.Sc. (Second) DEGREE EXAMINATION, MAY – 2019

Second Year

MATHEMATICS

Analytical Number Theory and Graph Theory

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) Show that in a complete graph with  $n$  vertices there are  $(n-1)/2$  edge – disjoint Hamiltonian circuits if  $n$  is an odd number such that  $n \geq 3$ .  
b) Draw a graph that has a Hamiltonian path but does not have a Hamiltonian circuit.
- Q2)** a) Show a graph with  $n$  vertices,  $(n-1)$  edges and no circuits is connected.  
b) Show that every tree has either one or two centers.
- Q3)** a) Show that every circuit has an even number of edges in common with any cut-set.  
b) Show that  $v$  in a connected graph  $G$  is a cut vertex if and only if there exist two vertices  $x$  and  $y$  in  $G$  such that every path between  $x$  and  $y$  passes through  $v$ .
- Q4)** Show that the complete graph of five vertices is non planar.
- Q5)** a) Show that the ring sum of two circuits in a graph  $G$  is either a circuit or an edge disjoint union of circuit.  
b) Write a note on the vector space associated with a graph  $G$ .



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ASSIGNMENT - 1

M.Sc. (Second) DEGREE EXAMINATION, MAY-2019

Second Year

MATHEMATICS

Rings and Modules

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) Prove that a Boolean algebra becomes a complemented distributive lattice by defining  $a \vee b = (a' \wedge b)'$ ,  $1 = 0'$ .
- b) If  $\theta$  is a reflexive homomorphic relation on a ring, then prove that  $\theta$  is symmetric and transitive, hence a congruence relation.
- Q2)** a) If A and B are ideals of a ring R then prove that AB,  $A \cdot B$  and  $A \cdot B$  are ideals of R. Moreover, prove that
- $AR = A = RA$
  - $A \cdot R = A = R \cdot A$
  - $A \cdot A = R = A \cdot A$
  - $AB \subset A \cap B$
- b) Prove that every proper right ideal in a ring is contained in a maximal proper right ideal.
- Q3)** a) Verify that  $\text{Hom}_R(A, B)$  is an Abelian group.
- b) Prove that a module has a composition series if and only if it is both Artinian and Noetherian.
- Q4)** a) Prove that a proper ideal M of the commutative ring R is maximal if and only if  $\forall r \notin M \exists n \in R 1 - rn \in M$
- b) Prove that every ring is a sub direct product of sub directly irreducible rings.
- Q5)** a) Let B be a Boolean algebra and K be an ideal of B. Then prove that K is maximal if and only if K is a prime ideal.
- b) If R is any commutative ring, then prove that  $Q(R)$  is rationally complete.

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ASSIGNMENT - 2

M.Sc. (Second) DEGREE EXAMINATION, MAY– 2019

Second Year

MATHEMATICS

Rings and Modules

MAXIMUM : 30 MARKS

ANSWER ALL QUESTIONS

- Q1)** a) If  $A_R$  is an irreducible module, then prove that its ring of endomorphisms  $D = \text{Hom}_R(A, A)$  is a division ring.  
b) Prove that  $R$  is a Prime ring if and only if  $1 \neq 0$  and, for all  $\alpha \neq 0$  and  $b \neq 0$  in  $R$ , there exists  $r \in R$  such that  $arb \neq 0$ .
- Q2)** a) Prove that the Prime radical of  $R$  is the set of all strongly nilpotent elements.  
b) Prove that  $R$  is semiprimitive if and only if it is a sub direct product of primitive rings.
- Q3)** a) If  $R$  is semiprime and  $e^2 = e \in R$ , then prove that  $eR$  is a minimal right ideal if and only if  $eRe$  is a division ring.  
b) Prove that the radical of a right Artinian ring is nilpotent.
- Q4)** a) Prove that  $M_R$  is free if and only if it is isomorphic to a direct sum of copies of  $R_R$ .  
b) Prove that every module is isomorphic to a factor of a free module.
- Q5)** a) Prove that an Abelian group is injective if and only if it is divisible.  
b) If  ${}_R F$  is a free module then prove that  $F_R^*$  is injective.

