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M.Sc. (Final) DEGREE EXAMINATION, DECEMBER 2012.

Second Year

Mathematics

Paper I – TOPOLOGY AND FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 100 marks

Answer any FIVE questions choosing at least TWO from each of Part A and Part B.

PART – A

1. (a) Prove that every separable metric space is second countable.

   (b) Define a topological space. If T_1 and T_2 are two topologies on a non empty set X, then show that $T_1 \cap T_2$ is also a topology on X.

2. (a) State and prove Heine – Borel theorem.

   (b) Prove that any continuous mapping of a compact metric space into a metric space is uniformly continuous.

3. State and prove Ascoli's theorem.

4. State and prove the Tietze extension theorem.

5. (a) Prove that a one - to - one continuous mapping of compact space onto a Hausdorff space is a homeomorphism.

   (b) Let X be a topological space and A a connected subspace of X. with the usual notation, if B is a subspce of X such that $A \subseteq B \subseteq \overline{A}$, then show that B is connected.

PART – B

6. (a) Define a normed linear space. Let N be a non - zero normed linear space. Then show that N is a Banach space $\Leftrightarrow \{x : \|x\| = 1\}$ in complete.

(b) Let N and N' be normed linear space and T a linear transformation of N into N'. Then show that the following conditions on T re equivalent.

(i) T is bounded

(ii) T is continuous

(iii) T is continuous at the origin in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

7. (a) State and prove the uniform boundedness theorem.

(b) Show with the usual notation that the mapping $T \to T^*$ is a norm preserving mapping of $B(N)$ into $B(N^*)$ when $B(N)$ is the normed linear space of all operators
on N.

8. (a) State and prove Bessel's in equality.

(b) Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M and let d be the distance from x to M. Then show that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

9. (a) Prove that an operator T on H is self - adjoint iff (Tx, x) is real for all x.

(b) If T is an arbitrary operator on H and if α and β are scalars such that $|\alpha| = |\beta|$, then show with the usual notation that $\alpha T + \beta T^*$ in normal.

10. (a) If T is an operator on H, then show that the following conditions are equivalent to one another :

(i) $T^{*}T = I$

(ii) $(T_x, T_y) = (x, y) \forall$ x and y

(iii) $\|Tx\| = \|x\| \forall x$

(b) If P and Q are the projections on closed linear subspaces M and N of H, then show that :

$$M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$$

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Mathematics

Paper II – MEASURE AND FUNCTIONAL ANALYSIS

Time : Three hours Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) Prove that the set of all finite sequences from a countable ser is also countable.

   (b) Prove that between any two real numbers, there is a rational number.

2. (a) Prove that the interval (a, α) is measurable.

   (b) Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets. Let mE_1 be finite. Than show that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \underset{n\to\infty}{Lt}\, mE_n$.

3. (a) Let f and g be two measurable Real valued functions defined on the same domain. Prove that $f + g$ and fg are also measurable.

   (b) Let f be a function with measurable domain D. Show that f is measurable iff the functions of defined by $g(n) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

4. (a) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then show that it is measurable and $R\int_a^b f(x)dx = \int_a^b f(x)dx$.

   (b) If f and g are bounded measurable functions defined on a set E of finite measure, then show that if $f \le g$ a.e; then $\int_E f \le \int_E g$. Hence show that $\left|\int_E f\right| \le \int_E |f|$.

5. (a) State and prove bounded convergence theorem.

(b) If $\{f_n\}$ is a sequence of non negative measurable functions and $f_n(x) \to f(x)$ a.e on a set E, then show that $\int_E f \leq Lim \int_E f_n$.

6. (a) If f is bounded and measurable on $[a,b]$ and $F(x) = \int_a^x f(t)dt + f\,F(a)$, then show that $F'(x) = f(x)$ for almost all x in $[a,b]$.

(b) Show that the sum and difference of two absolutely continuous functions are also absolutely continuous.

7. (a) State and prove Holder inequality.

(b) Let g be an integrable function on $[0,1]$ and suppose that there is a constant μ such that $\left|\int f\,g\right| \leq \mu \|f\|_p$ for all bounded measurable functions *f*. Then with the usual notation show that g is in $\lfloor v$ and $\|g\|_v \leq \mu$.

8. (a) If μ is a complete measure and f is a measurable function, then show that $f = g$ a.e implies g in measurable.

(b) Let E be a measurable set such that $0 < \gamma E < \infty$. Then with the usual notation show that there is a positive set A contained in E with $\gamma A > 0$.

9. State and prove Radom – Nikodym theorem.

10. (a) Prove with the usual notation that the class B of μ^* measurable sets is a $\sigma-$ algebra.

(b) If $A \in R$ an algebra of sets and if $\{A_i\}$ is any sequence of sets in R such that $A \subset \bigcup_{i=1}^{\infty} Ai$, then show with the usual notation that

(i) $\mu A \leq \sum_{i=1}^{\infty} \mu Ai$ (ii) $\mu^* A = \mu A$.

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M.Sc. (Final) DEGREE EXAMINATION, DECEMBER 2012.

Second Year

Mathematics

Paper III — ANALYTICAL NUMBER THEORY AND GRAPH THEORY

Time : Three hours Maximum : 100 marks

Answer any FIVE out of Ten questions.

Selecting atleast 2 from each Part.

PART A

1. (a) If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then

$$\sum_{y<n\le x} f(x) = \int_0^x f(t)\,dt + \int_y^x (t-[t])f'(t)\,dt + f(x)([x]-x) - f(y)([y]-y)$$

(b) For $x \ge 1$ prove that

$$\sum_{n\le x} d(n) = x\log x + (2(-1)\log x + o(\sqrt{x})).$$

2. (a) For $x > 1$ prove that

$\sum_{n\le x} \phi(n) = \frac{3}{\pi^2}x^2 + o(x\log x)$ and the average order of $\phi(n)$ is $\frac{3x}{\pi^2}$.

(b) For every $x \ge 1$, prove that

$[x]! = \prod_{p\le x} p^{\alpha(p)}$ where the product is extended over all primes $p \le x$ and $\alpha(p) = \sum_{n=1}^{\infty}\left[\frac{x}{p^n}\right]$.

3. (a) For $x \ge 2$ prove that

$\theta(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt$ and $\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t}\,dt$.

(b) Prove that the following relations are basically equivalent :

(i) $\lim_{x\to\infty}\frac{\pi(x)\log x}{x}=1$ (ii) $\lim_{x\to\infty}\frac{\theta(x)}{x}=1$ (iii) $\lim_{x\to\infty}\frac{\psi(x)}{x}=1$.

4. (a) For all $x\geq 1$ prove that

$$\sum_{n\leq x}\wedge(n)\left[\frac{x}{n}\right]=x\log x+o(x).$$

(b) Prove that there is a constant A such that

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+A+o\left(\frac{1}{\log x}\right) \text{ for all } x\geq 2.$$

PART B

5. (a) Prove that a simple graph with n-vertices k-components can have atmost $(n-k)(n-k+1)/2$ edges.

(b) In a connected graph G with exactly $2k$ odd vertices, there exist k-edge disjoint subgroups such that they together contain all edges of G and that each is a unicursal graph.

6. (a) Prove that in a complete graph with n-vertices there are $\frac{(n-1)}{2}$ edge disjoint Hamiltonian circuits, if n is odd number ≥ 3.

(b) Discuss the travelling salesman problem.

7. (a) Prove that any connected graph with n-vertices and $(n-1)$ edges is a true.

(b) Prove that A graph is a tree if one only if it is minimally connected.

8. (a) Prove that every circuit has an even number of edges is common with any cut set.

(b) Prove that the ring sum of any two cuts in a graph G is either a third cut or an edge disjoint union of cut sets.

9. (a) Prove that the complete graph of five vertices is non planar.

(b) Prove that A connected planar graph with n-vertices and e edges has $e-n+2$ regions.

10. (a) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge disjoint union of circuits.

(b) Under the ring sum operation $\oplus$, prove that the set consisting of all the circuits and the edge-disjoint unions of circuits (including the null set ϕ) is an abelian group.

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Second Year

Mathematics

Paper VI — RINGS AND MODULES

Time : Three hours | Maximum : 100 marks

Answer any FIVE questions.

All questions carry equal marks.

1. (a) If $\phi : R \to S$ is a homomorphism then show that there exists a congruence relation θ on R, an epimorphism $\pi : R \to R/\theta$ and a morphism $K : R|\theta \to S$ such that $\phi = K \circ \pi$.

   (b) Show that the congruence relations on a ring form a complete Lattice under inclusion.

2. (a) Show that the ideals in a ring form a complete modular Lattice under inclusion.

   (b) If A, B and C are additive subgroups of R show that $(AB)C = A(BC)$ and $AB \subset C \Leftrightarrow A \subset C : B$, $AB \subset C \Leftrightarrow B \subset A : C$.

3. (a) Show that a module is Noetherian if and only if every submodule is finitely generated.

   (b) Let B be a submodule of A_R. Show that A is Artinian if and only if B and $A|B$ are Artinian.

4. (a) Show that every proper ideal M of a commutative ring R is maximal if and only if for all $r \in M$ there exists $x \in R$ such that $1 - rx \in M$.

   (b) Show that the proper ideal P of the commutative ring R is prime if and only if for all elements a and b, $ab \in P$ implies $a \in P$ or $b \in P$.

5. (a) Show that every maximal ideal in a commutative ring is prime.

(b) Show that the radical of R consists of all elements $r \in R$ such that $1 - rx$ is a unit for all $x \in R$.

6. (a) Show that a ring R is primitive if and only if there exists a faithful irreducible module A_R.

(b) Prove that every primitive ideal is prime.

7. (a) Show that the prime radical of R is the set of all strongly nilpotent elements.

(b) Show that the radical of R is the set of all $r \in R$ such that $1 - rs$ is right invertible for all $s \in R$.

8. State and prove Wedderburn-Artin theorem.

9. (a) Show that every free module is projective.

(b) Show that every module is isomorphic to a factor of a free module.

10. (a) Show that M_R is injective if and only if for every right ideal K of R and every $\phi \in \mathrm{Hom}_R(KM)$, there exists an $m \in M$ such that $\phi k = mk$ for all $k \in K$.

(b) Show that an abelian group is injective if and only if it is divisible.

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