

ASSIGNMENT - 1, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - I : TOPOLOGY AND FUNCTIONAL ANALYSIS

Maximum : 30 MARKS

Answer ALL questions.

- 1) a) Let X be a topological space and A a subset of X . Then show that
 - i) $\bar{A} = A \cup D(A)$; and
 - ii) A is closed $\Leftrightarrow A \supseteq D(A)$.
 - b) Let X be a second countable space. Then show that any open base for X has a countable sub class which is also an open base.
 - 2) a) Prove that a topological space is compact if every basic open cover has a finite sub cover.
 - b) State and prove the Heine-Borel theorem.
 - 3) a) State and prove Tychonoff's theorem.
 - b) Prove that every sequentially compact metric space is compact.
 - 4) a) Prove that the product of any non-empty class of Hausdorff space is a Hausdorff space.
 - b) Prove that the range of a continuous real function defined on a connected space is an interval.
 - 5) State and prove the Urysohn imbedding theorem.
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ASSIGNMENT - 2, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - I : TOPOLOGY AND FUNCTIONAL ANALYSIS

Maximum : 30 MARKS

Answer ALL questions.

- 1) Let N and N' be normed linear spaces and T is a linear transformation of N into N' . Then show that the following on T are equivalent to one another.
- T is continuous.
 - T is continuous at the origin, in the sense that $x_n \rightarrow 0 \Rightarrow T(x_n) \rightarrow 0$;
 - There exists a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$;
 - If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N , then its image $T(S)$ is a bounded set in N' .
- 2) a) State and prove the open mapping theorem.
b) State and prove the uniform boundedness theorem.
- 3) a) State and prove Bessel's inequality.
b) If M is a proper closed linear subspace of a Hilbert space H , then show that there exists a non-zero vector z_0 in H such that $z_0 \perp M$.
- 4) a) If T is an operator on H for which $(Tx, x) = 0$ for all x , then prove that $T = 0$.
b) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then show that $N_1 + N_2$ and $N_1 N_2$ are normal.

- 5) a) If P is the projection on a closed linear subspace M of H , then show that M is invariant under an operator $T \Leftrightarrow TP = PTP$.
- b) If P and Q are the projections on closed linear subspaces M and N of H , then show that $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$.
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ASSIGNMENT - 1, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - II : MEASURE AND FUNCTIONAL ANALYSIS

Maximum : 30 MARKS

Answer ALL questions.

- 1) a) State the axiom of Archimedes. Show that between any two real numbers x and y there is a rational number r such that $x < r < y$.
- b) Prove that the set of all finite sequences from a countable set is also countable.
- 2) Prove that the outer measure of an interval is its length.
- 3) a) Prove that the interval (a, ∞) is measurable.
- b) Let $E \subset [0,1)$ be a measurable set. Then show that for each $y \in [0,1)$ the set $E + y$ is measurable and $m(E + y) = mE$.
- 4) a) Let f be a bounded function defined on $[a,b]$. If f is Riemann integrable on $[a,b]$, then show that it is measurable and $\mathbf{R} \int_a^b f(x) dx = \int_a^b f(x) dx$.
- b) State and prove Bounded convergence theorem.
- 5) a) If $\{f_n\}$ is a sequence of non negative measurable functions and $f_n(x) \rightarrow f(x)$ a-e on a set E , then show that $\int_E f \leq \liminf \int_E f_n$.
- b) State and prove monotone convergence theorem.
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ASSIGNMENT - 2, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - II : MEASURE AND FUNCTIONAL ANALYSIS

Maximum : 30 MARKS

Answer ALL questions.

- 1) Let f be an increasing real-valued function on the interval $[a, b]$. Then show that f is differentiable almost everywhere. The derivative f' is measurable, and $\int_a^b f'(x) dx \leq f(b) - f(a)$.
- 2) a) State and prove Hölder inequality.
b) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable.
- 3) a) State and prove Hahn decomposition theorem.
b) Prove that every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive.
- 4) a) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then show that there is a unique measurable extended real-valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \setminus B_\alpha$.
b) State and prove Lebesgue decomposition theorem.
- 5) a) Prove that the set function μ^* is an outer measure.
b) If μ^* is a Caratheodary outer measure with respect to Γ , then show that every function in Γ is μ^* -measurable.
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ASSIGNMENT - 1, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - III : ANALYTICAL NUMBER THEORY AND GRAPH THEORY

Maximum : 30 MARKS

Answer ALL questions.

- 1) a) If F has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then prove that

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t) dt + \int_y^x t - [t] f'(t) dt + f(x) [x] - x - f(y) [y] - y .$$

- b) Prove that if $x \geq 1$ we have $\sum_{n \leq x} \frac{1}{n} = \log x + c + O\left(\frac{1}{x}\right)$.

- 2) a) Prove that for $x \geq 1$ we have $\sum_{n \leq x} \mu(n) \left[\frac{x}{n} \right] = 1$ and $\sum_{n \leq x} \wedge(n) \left[\frac{x}{n} \right] = \log[x]!$

- b) Prove that for all $x \geq 1$ we have $\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1$, with equality holding only if $x < 2$.

- 3) a) Prove that for $x > 0$, we have, $0 \leq \frac{\psi(x)}{x} - \frac{v(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x} \log 2}$.

- b) State and prove Selberg's asymptotic formula.

- 4) a) Prove that for $n \geq 1$, the n^{th} prime p_n satisfies the inequalities $\frac{1}{6} n \log n < p_n < 12 \left(n \log n + n \log \frac{12}{e} \right)$.

- b) Let F be a real or complex valued function defined on $(0, \infty)$ and let

$$G(x) = \log x \sum_{n \leq x} F\left(\frac{x}{n}\right) \text{ then prove that } F(x) \log x + \sum_{n \leq x} F\left(\frac{x}{n}\right) \wedge(n) = \sum_{d \leq x} \mu(d) G\left(\frac{x}{d}\right).$$

- 5) a) Show that in a connected graph G with exactly $2K$ odd vertices, there exist K edge-disjoint sub graphs such that they together contain all edges of G and that each is a unicursal graph.
- b) Show that if a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.
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ASSIGNMENT - 2, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - III : ANALYTICAL NUMBER THEORY AND GRAPH THEORY

Maximum : 30 MARKS

Answer ALL questions.

- 1) a) Prove that a connected graph G is an Euler graph if and only if it can be decomposed into circuits.
b) Prove that in a complete graph with n vertices there are $\frac{(n-1)}{2}$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3 .
- 2) a) Prove that a graph is a tree if and only if it is minimally connected.
b) Show that Every connected graph has at least one spanning tree.
- 3) a) Prove that Every circuit has an even number of edges in common with any cut-set.
b) Prove that if G_1 and G_2 are two 1-isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 equals the nullity of G_2 .
- 4) a) Prove that a connected planar graph with n vertices and e edges has $e-n+2$ regions.
b) Prove that a necessary and sufficient condition for two planar graphs G_1 and G_2 to be duals of each other is as follows: There is a one to-one correspondence between the edges in G_1 and the edges in G_2 such that a set of edges in G_1 forms a circuit if and only if the corresponding set in G_2 forms a cut-set.

- 5) a) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.
- b) Prove that the set of circuit vectors corresponding to the set of fundamental circuits, with respect to any spanning tree, forms a basis for the circuit subspace W_r .
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ASSIGNMENT - 1, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - IV : RINGS AND MODULES

Maximum : 30 MARKS

Answer ALL questions.

- 1) a) Show that in any ring the following identities hold :
- $$a0 = 0 = 0a, \quad (-a)(-b) = ab.$$
- b) If ϕ is a homomorphism of a ring R into another ring, then prove that $\phi R \cong R/\phi^{-1}(0)$, Where ϕR is called the image, $\phi^{-1}0 = \{r \in R \mid \phi(r) = 0\}$, $\ker \phi$.
- 2) Prove that the following statements are equivalent :
- a) R is isomorphic to a finite direct product of rings $R_i (i=1, 2, \dots, n)$.
- b) There exist central orthogonal idempotents $e_i \in R$ such that $1 = \sum_{i=1}^n e_i$ and $e_i R \cong R_i$.
- c) R is a finite direct sum of ideals $k_i \cong R_i$.
- 3) a) Let C be a submodule of A_R . Prove that every submodule of A/C has the form B/C where $C \subseteq B \subseteq A$.
- b) Prove that a finite direct product of modules is Artinian if and only if each factor is Artinian.
- 4) a) Prove that every maximal ideal in a commutative ring is prime.
- b) If r is nilpotent. Show that $1-r$ is a unit.
- 5) a) If R is a commutative ring then prove that $Q(R)$ is regular if R is semiprime.
- b) If R is a Boolean ring, then prove that $Q(R)$ is a Boolean ring.
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ASSIGNMENT - 2, DEC - 2016.

M.Sc. (FINAL) SECOND YEAR DEGREE

MATHEMATICS

PAPER - IV : RINGS AND MODULES

Maximum : 30 MARKS

Answer ALL questions.

- 1) a) Prove that the prime radical of R is the set of all strongly Nilpotent elements.
b) Prove that the radical is an ideal and $R/\text{Rad } R$ is semiprimitive.
 - 2) a) If B is a submodule of A and C is maximal among the submodules of A such that $B \cap C = 0$, then prove that $B+C$ is large.
b) Prove that $\text{Rad } A = 0$ if $L(A)$ is complemented.
 - 3) a) Prove that in a right Artinian ring, the radical is the largest nilpotent ideal.
b) Prove that every finitely generated right ideal is principal in a regular ring.
 - 4) a) Prove that every free module is projective.
b) Prove that every module is isomorphic to a factor of free module.
 - 5) a) Prove that an abelian group is injective if and only if it is divisible.
b) Prove that there is a canonical monomorphism of M into $(M^*)^*$.
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